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Distributed delayed stabilization of Korteweg-de Vries-Burgers equation under sampled in space measurements

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***Abstract*— This article deals with distributed delayed stabilization of 1-D Korteweg-de Vries-Burgers (KdVB) equation under sampled in space measurements. The delay may be uncertain, but bounded by a known upper bound. On the basis of spatially distributed point measurements, we construct a regionally stabilizing controller applied through distributed in space shape functions. The existing Lyapunov-Krasovskii functionals for heat equation that depend on the state derivative are not applicable to KdVB equation because of the third spatial derivative. We suggest a new Lyapunov-Krasovskii functional that leads to regional stability conditions of the closed-loop system in terms of linear matrix inequalities (LMIs). By solving these LMIs, an upper bound on delay that preserves regional stability can be found, together with an estimate on the set of initial conditions starting from which the state trajectories of the system are exponentially converging to zero. This estimate includes a priori Lyapunov-based bounds on the solutions of the open-loop system on the initial time interval of the length of delay.**

I. INTRODUCTION

In recent decades, Korteweg-de Vries-Burgers (KdVB) equation has drawn a lot of attention as a nonlinear model of long waves in shallow water in a rectangular channel in which the effects of dispersion, dissipation and nonlinearity are present (see e.g. [5]). Without the diffusion term, KdVB equation becomes KdV equation, which has been proposed as a model of waves on shallow water surfaces. In recent decades, control of KdV and KdVB equations has been recently extensively studied by researchers. Regional boundary stabilization of KdV equation via the backstepping method by the state feedback [4] or output feedback controllers [33] has been considered. Global distributed stabilization [16] or boundary stabilization [31] of KdVB equation was suggested. For practical implementation of controllers for KdVB equation, it is important to ensure their robustness with respect to small input delay. In this work, we study distributed stabilizing controllers for KdVB equation suffering from uncertain input delays.

Stabilization of systems described by PDEs or a cascade of ODE-PDE subject to state/input / output time-delay or disturbance has been studied (see e.g. [17]- [27]). In [28] input delay compensation was suggested. In [14] and [24], delay-independent stabilization of heat equations with state delay was considered. Robustness with respect to small input/output time-varying delay of semilinear heat equation with globally Lipschitz nonlinearity was studied in [9], [10], where distributed control under point and averaged measurements was introduced. In [9], [10] global stability conditions were derived by using Lyapunov-Krasovskii functionals that depended on the state derivative via the descriptor method [7], [8]. In [1] similar control laws for reaction-diffusion equation were suggested, however, effects of input delay were not studied.

The present paper aims at introducing a distributed control of KdVB equation under point measurements with respect to constant input delay. Without delay, sufficient LMI conditions for global stabilization of KdVB equation are the same as for diffusion-reaction equations derived in [9] and [10]. However, analysis of stability of KdVB equation with delay is not a trivial task since the existing Lyapunov-Krasovskii functionals from [9]–[11] that depend on the state derivative are not applicable. We introduce a novel Lyapunov functional that depends on the state only and leads to LMI-based conditions of the closed-loop system. By solving these LMIs, an upper bound on delay that preserves regional stability can be found together with an estimate on the domain of attraction. As suggested for the case of ODEs in [30], our estimate on the domain of attraction is based on the Lyapunov-based bounds of the solutions on the initial time interval (of the length of delay), where the system is open-loop.

Notation. The Sobolev space $H^k(0,1)$ with $k \in Z$ is defined as $H^k(0,1) = \{z : D^\alpha z \in L^2(0,1),\ \forall\, 0 \le |\alpha| \le k\}$ with norm $\|z\|_{H^k} = \{\sum\limits_{0\le|\alpha|\le k} \|D^\alpha z\|^2_{L^2}\}^{\frac{1}{2}}$. For a square matrix P the notation $P > 0$ indicates that P is symmetric and positive-definite, the symbol $*$ denotes its symmetric elements.

Lemma 1. *(Wirtinger inequality [38]): For $a < b$, let $g \in H^1(a,b)$ be a scalar function with $g(a) = 0$ or $g(b) = 0$. Then*

$$\int_a^b g^2(x)dx \le \frac{4(b-a)^2}{\pi^2} \int_a^b \left[\frac{dg(x)}{dx}\right]^2 dx.$$

Lemma 2. *(Poincaré inequality [34]): For $a < b$, let $g \in H^1(a,b)$ be a scalar function with $\int_a^b g(x)dx = 0$. Then*

$$\int_a^b g^2(x)dx \le \frac{(b-a)^2}{\pi^2} \int_a^b \left[\frac{dg(x)}{dx}\right]^2 dx.$$

The following extension of Sobolev inequality will be useful:

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Lemma 3. *Let $z(x) \in H^1(0,1)$ be a scalar function. Then*

$$\max_{x\in[0,1]} z^2(x) \leq 2\int_0^1 z^2(x)dx + \int_0^1 z_x^2(x)dx.$$

Proof: Since $z(\cdot) \in H^1(0,1)$ implies $z(\cdot) \in C[0,1]$ (cf. [3], [35]), by mean value theorem, there exists $c \in (0,1)$ such that $z(c) = \int_0^1 z(x)dx$. Then, by integration by parts and further application of Jensen's and Young's inequalities, for all $x_1 \in [0,1]$ we have

$$\begin{aligned} z^2(x_1) &= z^2(c) + 2\int_c^{x_1} z(x)z_x(x)dx \\ &= \left[\int_0^1 z(x)dx\right]^2 + 2\int_c^{x_1} z(x)z_x(x)dx \\ &\leq 2\int_0^1 z^2(x)dx + \int_0^1 z_x^2(x)dx. \end{aligned}$$

II. System Description

Inspired by [6], we consider the following KdVB equation involving both instability and dissipation under the periodic boundary conditions:

$$\begin{cases} z_t(x,t) + z(x,t)z_x(x,t) - \beta z_{xx}(x,t) - \lambda z(x,t) \\ +z_{xxx}(x,t) = \sum\limits_{j=1}^N b_j(x)u_j(t-h),\ 0<x<1,\ t\geq 0, \\ z(0,t) = z(1,t),\ z_x(0,t) = z_x(1,t), \\ z_{xx}(0,t) = z_{xx}(1,t), \\ z(x,0) = z_0(x), \end{cases} \tag{1}$$

where $x \in (0,1)$, $\beta > 0$, $\lambda > 0$, $z(x,t)$ is the state of KdVB equation, $u_j(t) \in \mathbb{R}$, $(j = 1,2,\cdots,N)$ are the control inputs, $u_j(t) = 0$, $t < 0$, and $h > 0$ is a constant delay. The delay may be unknown, but bounded by a known bound $\bar{h} > 0$. For sufficiently large $\lambda > 0$, the open-loop system (with $u_j(t) \equiv 0$) may be unstable.

As in [1], [9], [10], [32], we assume that the points $0 = x_0 < x_1 < \cdots < x_N = 1$ divide $[0,1]$ into N intervals $\Omega_j = [x_{j-1}, x_j)$ that are upper bounded by Δ:

$$0 < x_j - x_{j-1} = \Delta_j \leq \Delta.$$

The control inputs $u_j(t)$ enter (1) through the shape functions $b_j(x)$ such that

$$\begin{cases} b_j(x) = 0,\ x \notin \Omega_j, \\ b_j(x) = 1,\ otherwise, \end{cases} \quad j = 1,\cdots,N. \tag{2}$$

We assume further that sensors provide point measurements of the state

$$y_j(t) = z(\bar{x}_j, t),\quad \bar{x}_j = \frac{x_{j-1}+x_j}{2},\ t > h. \tag{3}$$

We design a regionally stabilizing distributed controller

$$u_j(t) = \begin{cases} -\mu y_j(t),\ j = 1,\cdots,N, & t > h, \\ 0, & t \leq h, \end{cases} \tag{4}$$

where μ is a positive constant to be determined later, and $y_j(t)$ is given by (3).

Denote the characteristic function of the time interval $[0,h]$ by $\chi_{[0,h]}(t)$. Under the control law (4), the closed-loop system becomes

$$\begin{cases} z_t(x,t) + z(x,t)z_x(x,t) - \beta z_{xx}(x,t) - \lambda z(x,t) + z_{xxx}(x,t) \\ = -\mu \sum\limits_{j=1}^N b_j(x)(1-\chi_{[0,h]}(t))[z(x,t-h) - f_j(x,t-h)], \\ \qquad\qquad 0<x<1,\ t\geq 0, \\ z(0,t) = z(1,t),\ z_x(0,t) = z_x(1,t), \\ z_{xx}(0,t) = z_{xx}(1,t), \\ z(x,0) = z_0(x), \end{cases} \tag{5}$$

where

$$f_j(x,t-h) = \int_{\bar{x}_j}^x z_\zeta(\zeta, t-h)d\xi. \tag{6}$$

A. Well-posedness of (5) subject to (6)

Define $H^1_{per}(0,1) = \{g \in H^1(0,1) : g(0) = g(1)\}$, and $\|g\|^2_{H^1_{per}} = P_1\int_0^1 g^2(x)dx + P\int_0^1 [g'(x)]^2dx$. Here P_1 and P are positive constants that are related to the Lyapunov-Krasovskii functional (see (8) below) that will be used for stability analysis. It is obvious that $H^1_{per}(0,1)$ is a subspace of Sobolev space $H^1(0,1)$. Moreover, the norm $\|\cdot\|_{H^1_{per}(0,1)}$ is equivalent to $\|\cdot\|_{H^1(0,1)}$.

By the arguments of [29], the well-posedness of the system (5) subject to (6) can be obtained:

Lemma 4. *Assume that the initial value $z_0 \in H^3(0,1) \cap H^1_{per}(0,1)$ satisfies the compatible conditions:*

$$z_0'(0) = z_0'(1),\ z_0''(0) = z_0''(1). \tag{7}$$

Then, for all $T > 0$, there exists a unique solution to the system (5) subject to (6) from the class

$$\begin{aligned} &z \in C(0,T; H^1_{per}(0,1)), \\ &z_t \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1_{per}(0,1)). \end{aligned}$$

III. Regional stability of the delayed KdVB equation

Consider the following Lyapunov-Krasovskii functional:

$$V(t) = V_{aug} + V_P + V_R + V_Q + V_S + V_W, \tag{8}$$

where

$$\begin{aligned} V_{aug} &= \int_0^1 \theta^T \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \theta dx,\ \theta = \begin{bmatrix} z(x,t) \\ \int_{t-h}^t z(x,s)ds \end{bmatrix}, \\ V_P &= P\int_0^1 z_x^2(x,t)dx, \\ V_R &= R\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}(s+h-t)z^2(x,s)dsdx, \\ V_Q &= Q\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}z^2(x,s)dsdx, \\ V_S &= S\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}(s+h-t)z_x^2(x,s)dsdx \\ V_W &= W\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}z_x^2(x,s)dsdx. \end{aligned}$$

Here $P > 0$, $R > 0$, $Q > 0$, $S > 0$, and $W > 0$. Moreover, we assume that for some constant $\gamma > 0$ the following LMI holds:

$$\begin{bmatrix} P_1 - \gamma & P_2 \\ * & P_3 + Q\dfrac{e^{-2\delta h}}{h} \end{bmatrix} > 0. \tag{9}$$

Due to Jensen's inequality [12]

$$\begin{aligned} V_Q &\geq Qe^{-2\delta h}\int_0^1\int_{t-h}^t z^2(x,s)dsdx \\ &\geq Q\frac{e^{-2\delta h}}{h}\int_0^1\left[\int_{t-h}^t z(x,s)ds\right]^2dx. \end{aligned}$$

LMI (9) guarantees the positivity of the Lyapunov-Krasovskii functional:

$$V(t)\geq \gamma\int_0^1 z^2(x,t)dx + P\int_0^1 z_x^2(x,t)dx. \tag{10}$$

Theorem 1. *Consider the system (5) subject to (6). Given positive scalars Δ, $\mu>\lambda$, h, δ, and positive tuning parameters $\delta_1>\lambda$, $C>0$ and $C_1>0$. Let there exist scalars $\gamma\geq 2$, $P_1>0$, $P_2<0$, $P>1$, $Q>0$, $R>0$, $S>0$, $W>0$, $\alpha>0$, $P_3\in\mathbb{R}$ and $\nu\in\mathbb{R}$ such that (9),*

$$-We^{-2\delta h}+\alpha\leq 0, \tag{11}$$

$$\Xi\big|_{z=C_1}\leq 0,\ \Xi\big|_{z=-C_1}\leq 0, \tag{12}$$

$$\Lambda\big|_{z=C_1}\leq 0,\ \Lambda\big|_{z=-C_1}\leq 0, \tag{13}$$

$$\Phi\big|_{z=C_1}\leq 0,\ \Phi\big|_{z=-C_1}\leq 0, \tag{14}$$

hold, where

$$\Xi=\begin{bmatrix} -2P_1(\delta_1-\lambda) & 0 & \nu \\ * & \Xi_{22} & Pz \\ * & * & -2P\beta \end{bmatrix}, \tag{15}$$

$$\Xi_{22}=-2P_1\beta-2P(\delta_1-\lambda)+2\nu,$$

$$\Phi=\begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & \nu & \Phi_{15} & 0 & P_1\mu \\ * & \Phi_{22} & 0 & P\mu & \Phi_{25} & 0 & 0 \\ * & * & \Phi_{33} & Pz & -P_2z & -P_2\beta & 0 \\ * & * & * & \Phi_{44} & 0 & P_2 & 0 \\ * & * & * & * & \Phi_{55} & 0 & -P\mu \\ * & * & * & * & * & \Phi_{66} & P_2\mu \\ * & * & * & * & * & * & \Phi_{77} \end{bmatrix}, \tag{16}$$

$$\Lambda=\begin{bmatrix} \Lambda_{11} & -P_2 & 0 & \nu & \Lambda_{15} & 0 \\ * & \Lambda_{22} & 0 & 0 & -P_3 & 0 \\ * & * & \Lambda_{33} & Pz & -P_2z & -P_2\beta \\ * & * & * & \Lambda_{44} & 0 & P_2 \\ * & * & * & * & \Lambda_{55} & 0 \\ * & * & * & * & * & \Lambda_{66} \end{bmatrix}, \tag{17}$$

$$\begin{aligned} &\Phi_{11}=2P_2+2P_1\lambda+Rh+Q+2\delta P_1, \\ &\Lambda_{11}=\Phi_{11}-2\delta_1P_1, \\ &\Phi_{12}=-P_1\mu-P_2, \\ &\Phi_{15}=\Lambda_{15}=P_3+2\delta P_2+P_2\lambda, \\ &\Phi_{22}=\Lambda_{22}=-Qe^{-2\delta h}, \\ &\Phi_{25}=-P_2\mu-P_3, \\ &\Phi_{33}=-2P_1\beta+2P\lambda+2\nu+Sh+2\delta P+W, \\ &\Lambda_{33}=\Phi_{33}-2\delta_1P, \\ &\Phi_{44}=\Lambda_{44}=-2P\beta, \\ &\Phi_{55}=\Lambda_{55}=-Re^{-2\delta h}\frac{1}{h}+2\delta P_3, \\ &\Phi_{66}=\Lambda_{66}=-Se^{-2\delta h}\frac{1}{h}, \\ &\Phi_{77}=-\alpha\frac{\pi^2}{\Delta^2}. \end{aligned} \tag{18}$$

Denote

$$\begin{aligned} M=\max\ &\left\{\left(P_1+2P_2h+P_3h^2+(\frac{Rh^2}{2}+Qh)\right)P_1^{-1},\right. \\ &\left.(P+\frac{Sh^2}{2}+Wh)P^{-1}\right\}+(e^{2\delta_1h}-1). \end{aligned}$$

If

$$MC^2<C_1^2, \tag{19}$$

then for any initial state $z_0\in H^3(0,1)\cap H^1_{per}(0,1)$ satisfying the compatible conditions (7) and $\|z_0\|_{H^1_{per}}<C$, system (5) subject to (6) possesses a unique solution. Moreover, the solution of (5) subject to (6) satisfies

$$V(t)\leq Me^{-2\delta(t-h)}\left[P_1\int_0^1 z_0^2(x)dx+P\int_0^1[z_0'(x)]^2dx\right] \tag{20}$$

for all $t\geq h$.

Proof: We divide the proof into three parts.

Step 1: By arguments of [8], [30], we first derive a simple bound on $V(h)$ in terms of z_0 such that $V(h)<C_1^2$. Since the solution of the system (5) subject to (6) does not depend on the values of $z(x,t)$ for $t<0$, we redefine the initial condition to be constant:

$$z(x,t)=z_0(x),\ t\leq 0. \tag{21}$$

Due to (21), we have

$$V_{aug}(0)=[P_1+2P_2h+P_3h^2]\int_0^1 z_0^2(x)dx.$$

Then

$$\begin{aligned} V(0)\ &\leq [P_1+2P_2h+P_3h^2+\tfrac{Rh^2}{2}+Qh]\int_0^1 z_0^2(x)dx \\ &+(P+\tfrac{Sh^2}{2}+Wh)\int_0^1[z_0'(x)]^2dx. \end{aligned} \tag{22}$$

We consider

$$V_0(t)=P_1\int_0^1 z^2(x,t)dx+P\int_0^1 z_x^2(x,t)dx. \tag{23}$$

Given $\delta>0$. Assume that there exists $\delta_1>0$ such that

$$\dot V_0(t)-2\delta_1V_0(t)\leq 0,\ t\in[0,h], \tag{24}$$

$$\dot V(t)+2\delta V(t)-2\delta_1V_0(t)\leq 0,\ t\in[0,h], \tag{25}$$

then

$$V_0(t)\leq e^{2\delta_1t}V_0(0),\ t\in[0,h], \tag{26}$$

$$V(t)\leq e^{-2\delta t}V(0)+(e^{2\delta_1t}-1)V_0(0),\ t\in[0,h]. \tag{27}$$

Substituting (22) into the inequality (27), together with the condition (19), we have

$$V(h)\leq M\left[P_1\int_0^1 z_0^2(x)dx+P\int_0^1[z_0'(x)]^2dx\right]<C_1^2, \tag{28}$$

if

$$\|z_0\|^2_{H^1_{per}}=P_1\int_0^1 z_0^2(x)dx+P\int_0^1[z_0'(x)]^2dx<C^2. \tag{29}$$

Step 2: We show next the LMIs (9), (12) and (13) guarantee that (24) and (25) hold.

Differentiating V_{aug} along (5), for $t \in [0,h]$ we obtain

$$\begin{array}{l}\dot{V}_{aug} = -2P_1\beta\int_0^1 z_x^2(x,t)dx + 2P_1\lambda\int_0^1 z^2(x,t)dx \\ -2P_2\int_0^1 z(x,t)z_x(x,t)\int_{t-h}^t z(x,s)dsdx \\ -2P_2\beta\int_0^1 z_x(x,t)\int_{t-h}^t z_x(x,s)dsdx \\ +2P_2\int_0^1 z_{xx}(x,t)\int_{t-h}^t z_x(x,s)dsdx \\ +2P_2\lambda\int_0^1 z(x,t)\int_{t-h}^t z(x,s)dsdx \\ +2P_2\int_0^1 z(x,t)[z(x,t)-z(x,t-h)]dx \\ +2P_3\int_0^1\int_{t-h}^t z(x,s)ds[z(x,t)-z(x,t-h)]dx.\end{array} \tag{30}$$

We have

$$\begin{array}{rl}\dot{V}_P & = -2P\beta\int_0^1 z_{xx}^2(x,t)dx + 2P\lambda\int_0^1 z_x^2(x,t)dx \\ & +2P\int_0^1 z_{xx}(x,t)z_x(x,t)z(x,t)dx,\end{array} \tag{31}$$

$$\dot{V}_Q + 2\delta V_Q = Q\int_0^1 z^2(x,t)dx - Q\int_0^1 e^{-2\delta h}z^2(x,t-h)dx, \tag{32}$$

and

$$\dot{V}_W + 2\delta V_W = W\int_0^1 z_x^2(x,t)dx - W\int_0^1 e^{-2\delta h}z_x^2(x,t-h)dx. \tag{33}$$

Further by applying Jensen's inequality we obtain

$$\begin{array}{l}\dot{V}_R + 2\delta V_R \\ \leq Rh\int_0^1 z^2(x,t)dx - Re^{-2\delta h}\frac{1}{h}\int_0^1\left[\int_{t-h}^t z(x,s)ds\right]^2 dx,\end{array} \tag{34}$$

and

$$\begin{array}{l}\dot{V}_S + 2\delta V_S \\ \leq Sh\int_0^1 z_x^2(x,t)dx - Se^{-2\delta h}\frac{1}{h}\int_0^1\left[\int_{t-h}^t z_x(x,s)ds\right]^2 dx.\end{array} \tag{35}$$

Additionally,

$$2\nu\left[\int_0^1 z(x,t)z_{xx}(x,t)dx + \int_0^1 z_x^2(x,t)dx\right] = 0 \quad \forall \nu \in \mathbb{R}. \tag{36}$$

We add to $\dot{V}(t) + 2\delta V(t)$ the left-hand side of (36). Then, by taking into account (30)-(35), for $t \in [0,h]$ we arrive at

$$\begin{array}{l}\dot{V}(t) + 2\delta V(t) - 2\delta_1 V_0(t) \\ \leq \int_0^1 \psi^\top(x,t)\Lambda\psi(x,t)dx - We^{-2\delta h}\int_0^1 z_x^2(x,t-h)dx,\end{array} \tag{37}$$

where

$$\begin{array}{rl}\psi(x,t) = & \mathrm{col}\{z(x,t), z(x,t-h), z_x(x,t), z_{xx}(x,t), \\ & \int_{t-h}^t z(x,s)ds, \int_{t-h}^t z_x(x,s)ds\},\end{array} \tag{38}$$

and Λ is given by (17).

Similarly, differentiating $V_0(t)$ along (5) and adding (36), we have

$$\dot{V}_0(t) - 2\delta_1 V_0(t) = \int_0^1 \xi^\top(x,t)\Xi\xi(x,t)dx,\ t \in [0,h], \tag{39}$$

where $\xi(x,t) = \mathrm{col}\{z(x,t), z_x(x,t), z_{xx}(x,t)\}$ and Ξ is given by (15).

As in [36], first we assume that

$$z(x,t) \in (-C_1, C_1) \quad \forall x \in [0,1],\ \forall t \in [0,h]. \tag{40}$$

From (37) and (39), it follows that if $\Xi \leq 0$ and $\Lambda \leq 0$ for all $z \in (-C_1, C_1)$, then (24) and (25) hold. Matrices Ξ and Λ given by (15) and (17) are affine in z. Thus, $\Xi \leq 0$ and $\Lambda \leq 0$ for all $z \in (-C_1, C_1)$ if LMIs (12) and (13) are feasible. Therefore, (12) and (13) guarantee that (24) and (25) hold.

We prove next (40). If the LMI (9) is feasible, then Lemma 3 leads to

$$\begin{array}{l}\max_{0\leq x\leq 1}|z(x,t)|^2 \leq 2\int_0^1 z^2(x,t)dx + \int_0^1[z_x(x,t)]^2dx \\ \leq \gamma\int_0^1 z^2(x,t)dx + P\int_0^1[z_x(x,t)]^2dx. \\ \leq P_1\int_0^1 z^2(x,t)dx + P\int_0^1[z_x(x,t)]^2dx \\ = V_0(t) \qquad \forall t \in [0,h].\end{array} \tag{41}$$

Moreover, from (10), we have

$$\begin{array}{l}\max_{0\leq x\leq 1}|z(x,t)|^2 \leq \gamma\int_0^1 z^2(x,t)dx + P\int_0^1[z_x(x,t)]^2dx \\ \qquad \leq V(t) \qquad \forall t \in [0,h].\end{array} \tag{42}$$

Therefore, in a manner similar to the proof of [36], one can show that

$$V_0(t) < C_1^2, \qquad V(t) < C_1^2$$

for all $t \in [0,h]$.

Thus, (40) and consequently, (24) and (25) are true on $[0,h]$.

Step 3: We continue to find sufficient conditions in terms of LMIs to guarantee $\dot{V}(t) + 2\delta V(t) \leq 0$ for all $t > h$.

Differentiating V and integrating by parts, we obtain (32)-(35). For $t > h$, (30)-(31) become

$$\begin{array}{l}\dot{V}_{aug} = -2P_1\beta\int_0^1 z_x^2(x,t)dx + 2P_1\lambda\int_0^1 z^2(x,t)dx \\ -2P_1\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j} z(x,t)[z(x,t-h) - f_j(x,t-h)]dx \\ -2P_2\int_0^1 z(x,t)z_x(x,t)\int_{t-h}^t z(x,s)dsdx \\ -2P_2\beta\int_0^1 z_x(x,t)\int_{t-h}^t z_x(x,s)dsdx \\ +2P_2\int_0^1 z_{xx}(x,t)\int_{t-h}^t z_x(x,s)dsdx \\ +2P_2\lambda\int_0^1 z(x,t)\int_{t-h}^t z(x,s)dsdx \\ -2P_2\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j}[z(x,t-h) - f_j(x,t-h)]\int_{t-h}^t z(x,s)dsdx \\ +2P_2\int_0^1 z(x,t)[z(x,t)-z(x,t-h)]dx \\ +2P_3\int_0^1\int_{t-h}^t z(x,s)ds[z(x,t)-z(x,t-h)]dx,\end{array} \tag{43}$$

and

$$\begin{array}{rl}\dot{V}_P & = -2P\beta\int_0^1 z_{xx}^2(x,t)dx + 2P\lambda\int_0^1 z_x^2(x,t)dx \\ & +2P\int_0^1 z_{xx}(x,t)z_x(x,t)z(x,t)dx \\ & +2P\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j} z_{xx}(x,t)[z(x,t-h) - f_j(x,t-h)]dx.\end{array} \tag{44}$$

Lemma 1 yields

$$\begin{array}{l}\int_{x_{j-1}}^{x_j} f_j^2(x,t-h)dx = \int_{x_{j-1}}^{x_j}\left[\int_{\bar{x}_j}^x z_\zeta(\zeta,t-h)d\xi\right]^2 dx \\ \leq \frac{\Delta^2}{\pi^2}\int_{x_{j-1}}^{x_j} z_x^2(x,t-h)dx \quad \forall t > h.\end{array}$$

Hence,

$$\alpha\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[z_x^2(x,t-h) - \frac{\pi^2}{\Delta^2}f_j^2(x,t-h)\right]dx \geq 0 \tag{45}$$

holds for some constant $\alpha > 0$.

By adding to $\dot{V}(t) + 2\delta V(t)$ the equality (36), and using (32)-(35), (43)-(45), we obtain

$$\begin{aligned} &\dot{V} + 2\delta V \\ &\leq \dot{V} + 2\delta V + \alpha \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \left[z_x^2(x, t-h) - \tfrac{\pi^2}{\Delta^2} f_j^2(x, t-h) \right] dx \\ &\leq \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \eta^\top \Phi \eta dx - (We^{-2\delta h} - \alpha) \int_0^1 z_x^2(x, t-h) dx, \end{aligned} \tag{46}$$

where $\eta = \mathrm{col}\{z(x,t), z(x,t-h), z_x(x,t), z_{xx}(x,t), \int_{t-h}^{t} z(x,s)ds, \int_{t-h}^{t} z_x(x,s)ds, f_j(x,t-h)\}$.

Matrix Φ given by (16) is affine in z. Hence, $\Phi \leq 0$ for all $z \in (-C_1, C_1)$ if (14) is satisfied. Therefore, (11) and (14) guarantee $\dot{V}(t) + 2\delta V(t) \leq 0$, which implies

$$V(t) \leq e^{-2\delta(t-h)} V(h) \qquad \forall t \geq h. \tag{47}$$

Using (28) and (47), we obtain (20).

IV. Example

Consider the system (1) with parameters $\beta = 0.5$ and $\lambda = 15$. Here for the control law (4) with the point measurements, by using Yalmip we verify LMI conditions of Theorem 1 with $\mu = 20$, $\Delta = 0.1$, $\delta = 1$, $\delta_1 = 20$, $C = 0.044$, $C_1 = 0.05$. We find that the closed-loop system (5) subject to (6) preserves the exponential stability for $h \leq 0.00189$ for any initial values satisfying $\|z_0\|_{H^1_{per}} < 0.044$.

Next a finite difference method is applied to compute the state of the closed-loop system (5) subject to (6) to illustrate the effect of the proposed feedback control law (4) with the point measurements. We choose the same values of parameters and the initial condition $z_0(x) = 0.0025 \sin(2\pi x)$, $0 \leq x \leq 1$. Hence,

$$\|z_0\|^2_{H^1_{per}} = 387.3590\|z_0\|^2 + 5.5792\|z_0'\|^2 < 0.044^2.$$

The steps of space and time are taken as 0.05 and 10^{-7}, respectively. Simulation of solutions under the controller

$$u_j(t) = \begin{cases} -20 z(\bar{x}_j, t), & t > 0.00189, \\ 0, & 0 \leq t \leq 0.00189 \end{cases}$$

with $x_j - x_{j-1} = \Delta_j = \Delta = 0.1$, $\bar{x}_j = \dfrac{x_{j-1} + x_j}{2}$, $j = 1, \cdots, 10$, where the spatial domain is divided into ten subdomains, shows that the closed-loop system is exponentially stable (see Fig. 1). Enlarging the value of h until 0.01, we find that the solution starting from the same initial condition is unbounded (see Fig. 2). The simulations of the solutions confirm the theoretical results.

V. Conclusions

In this work, we studied distributed control of the KdVB equation in the presence of uncertain and bounded constant delay under the spatially distributed point measurements. By constructing a novel augmented Lyapunov function, we derived sufficient conditions ensuring that the closed-loop system is regionally stable. The future work will be devoted to the extension of the obtained results to the observer-based boundary control of nonlinear PDEs.

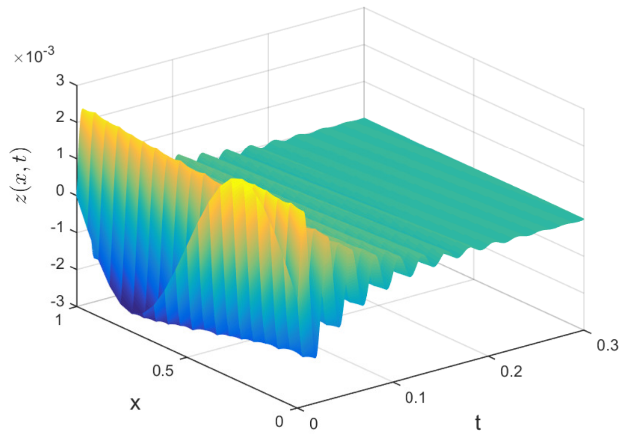

Fig. 1. State $z(x,t)$ with $h = 0.00189$ and $x_j - x_{j-1} = 0.1$ under the point measurements

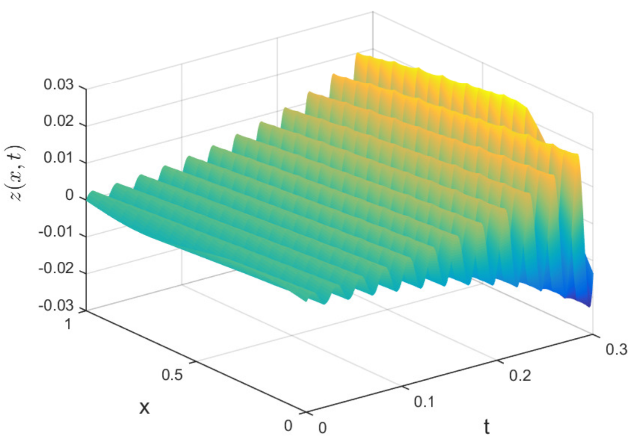

Fig. 2. State $z(x,t)$ with $h = 0.01$ and $x_j - x_{j-1} = 0.1$ under the point measurements

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