23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

Solvability of Dirichlet problem with Nonlinear Integro-differential Operator

Erhan Bayraktar[1] and Qingshuo Song[2]

Abstract— **This paper studies the solvability of a class of Dirichlet problem associated with a non-linear integro-differential operator. The main ingredient is the use of Perron's method together with the probabilistic construction of continuous supersolution via the identification of the continuity set of the exit time operators in the path space under Skorohod topology.**

I. Introduction and problem setup

A. Problem setup

Consider an equation of the form

$$F(u,x)+u(x)-\ell(x)=0,\ x\in O \tag{1}$$

with the boundary value

$$u(x)=g(x),\ x\in O^c. \tag{2}$$

In the above, the operator

$$F(u,x)=-\inf_{a\in[\underline{a},\overline{a}]}H(u,x,a)-\mathcal{I}(u,x)$$

is defined via operators given by

$\mathcal{I}(u,x)=\int_{\mathbb{R}^d}(u(x+y)-u(x)-Du(x)\cdot yI_{B_1}(y))\nu(dy)$;

$H(u,x,a)=\frac{1}{2}tr(A(a)D^2u(x))+b(a)\cdot Du(x)$ with $A(a)=\sigma'(a)\sigma(a)$.

In the above, $\underline{a}\le\overline{a}$ are given two real numbers, $\nu(\cdot)$ is a Levy measure on $\mathbb{R}^d$, $B_r(x)$ is a ball of radius r with center x, and $B_r=B_r(0)$ for simplicity. Recall that, we say ν is a Levy measure, if $\int_{\mathbb{R}^d}(1\wedge|y|^2)\nu(dy)<\infty$ holds. To simplify our presentation, we will use the following additional assumptions throughout the paper.

Assumption 1:
1) O is a connected open bounded set in $\mathbb{R}^d$.
2) $\sigma,b\in C^{0,1}(\mathbb{R})$; $\ell,g\in C_0(\mathbb{R}^d)$.
3) $\nu(dy)=\hat{\nu}(y)dy$ is a Levy measure satisfying $\hat{\nu}\in C_0(\mathbb{R}^d\setminus\{0\})$.

For some $\alpha\in(0,2)$, if ν is given by

$$\nu(dy)=\frac{dy}{|y|^{d+\alpha}},$$

then ν satisfies Assumption 1, and the integral operator is denoted by $\mathcal{I}(u,x)=-(-\Delta)^{\alpha/2}u(x)$ as convention. For convenience, we write $-(-\Delta)^0u=0$.

B. Literature review and an example

A function u is said to be a solution of Dirichlet problem (1)-(2), if $u\in C(\bar{O})$ satisfies (1) in the viscosity sense and $u=g$ on ∂O. It is worth to note that, as far as Dirichlet problem (1)-(2) concerned, one can generalize the boundary condition (2) by

$$\begin{aligned}&\max\{F(u,x)+u(x)-\ell(x),u-g\}\ge 0\\&\ge\min\{F(u,x)+u(x)-\ell(x),u-g\}\text{ on }O^c\end{aligned} \tag{3}$$

without loss of uniqueness in the viscosity sense.

In contrast to the (classical) Dirichlet problem (1)-(2), Dirichlet problem (1)-(3) is referred to a generalized Dirichlet problem. For the generalized Dirichlet problem without nonlocal operator, there were many excellent discussions on the solvability with the comparison principle and Perron's method, see for instance, [Barles and Perthame, 1988], [Barles and Perthame, 1990], [Barles and Burdeau, 1995], and Section 7 of [Crandall et al., 1992]. Recently, the solvability result has been extended to nonlinear equations associated to Integro-differential operators, see [Barles and Imbert, 2008], [Barles et al., 2008], [Alvarez and Tourin, 1996], [Topp, 2014], and the references therein.

Compared to the generalized Dirichlet problem, there are relatively less discussions available on the classical Dirichlet problem associated with the Integral operators in the aforementioned references. For the illustration purpose, we will use the following example, which will be used throughout the paper.

Example 1: Justify the the uniqueness and existence of the viscosity solution for Dirichlet problem given by, with $\alpha\in[0,2]$

$$|\partial_{x_1}u|+(-\Delta)^{\alpha/2}u+u-1=0,\ \forall x\in O=(-1,1)\times(-1,1) \tag{4}$$

with

$$u(x)=0,\ \forall x\in O^c.$$

A partial answer of Example 1 from the existing literature is given as this below:

- If $\alpha=0$, there is no solution. In fact, one can directly check that $u(x)=1-e^{-1+|x_1|}$ is the unique solution of the generalized Dirichlet problem, but not a solution of classical Dirichlet problem due to its loss of boundary at $\{(x_1,x_2):|x_2|=1,|x_1|<1\}$.
- If $\alpha\in[1,2]$, there is a unique solution by [Barles et al., 2008].
- If $\alpha\in(0,1)$, although there is unique solution of generalized Dirichlet problem by [Topp, 2014], it is

*It is an extended abstract for MTNS 2018. This work was supported by CityU SRG of Hong Kong (7004667).
[1]Department of Mathematics, University of Michigan, USA erhan@umich.edu
[2]Department ofMathematics, City University of Hong Kong, song.qingshuo@cityu.edu.hk

remained unanswered whether there is a solution of classical Dirichlet problem.

C. Work outline

This work focuses on the sufficient condition of the existence and uniqueness of the viscosity solution for Dirichlet problem of (1)-(2). Formally, the solution of (1)-(2), if it exists, is expected to be equal to the value function of a stochastic exit control problem, see for instance [Fleming and Soner, 2006]. However, a rigorous proof on the equivalence between the solution of (1)-(2) and the value function associated to exit control problem is not an easy task due to the lack of dynamic programming principle, see more discussions in [Bayraktar and Sirbu, 2013] for Hamilton-Jacobi-Bellman equation without non-local operator.

Alternatively, our approach here is to construct the subsolution and supersolution, and then the unique solvability follows by the comparison principle and Perron's method. The comparison principle and Perron's method are already available in [Barles and Imbert, 2008]. In this connection, we establishes the main result by constructing subsolution and supersolution using a particular controlled process and boundary data g. Finally, we emphasize that the proof relies on the continuity of the value function of the exit problem. In general, due to the non-local property, continuity of the value function up to a stopping time is much more delicate than the counterpart of the purely differential form. Indeed, by investigating the continuity of the exit mappings on path space under Skorohod metric, we conclude that the regularity of the boundary guarantees the continuity of the value function. This part is crucial for the main result, and as far as the solvability of Dirichlet problem concerned, the methodology is original to the best of our knowledge.

The contribution of this work is therefore the sufficient condition on the existence and uniqueness of the solution for (1)-(2). In particular, the sufficient condition requires the regularity of the boundary with respect to some controlled process. When Dirichlet problem is given by purely differential elliptic operators with C^2-smooth boundary, our result is consistent to Example 4.6 of [Crandall et al., 1992] and [Barles and Burdeau, 1995]. Nevertheless, it is also useful for Dirichlet problem when the regularity of the boundary is known for a nonsmooth domain. Back to Example 1, one can easily show that existence and uniqueness holds for any constant $\alpha \in (0, 2]$. It is also noted that existence and uniqueness still holds for $\alpha \in (0, 2]$ as long as the boundary satisfies exterior cone condition, and this is also a new result.

References

[Alvarez and Tourin, 1996] Alvarez, O. and Tourin, A. (1996). Viscosity solutions of nonlinear integro-differential equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(3):293–317.

[Barles and Burdeau, 1995] Barles, G. and Burdeau, J. (1995). The Dirichlet problem for semilinear second-order degenerate elliptic equations and applications to stochastic exit time control problems. *Comm. Partial Differential Equations*, 20(1-2):129–178.

[Barles et al., 2008] Barles, G., Chasseigne, E., and Imbert, C. (2008). On the Dirichlet problem for second-order elliptic integro-differential equations. *Indiana Univ. Math. J.*, 57(1):213–246.

[Barles and Imbert, 2008] Barles, G. and Imbert, C. (2008). Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(3):567–585.

[Barles and Perthame, 1988] Barles, G. and Perthame, B. (1988). Exit time problems in optimal control and vanishing viscosity method. *SIAM J. Control Optim.*, 26(5):1133–1148.

[Barles and Perthame, 1990] Barles, G. and Perthame, B. (1990). Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations. *Appl. Math. Optim.*, 21(1):21–44.

[Bayraktar and Sirbu, 2013] Bayraktar, E. and Sirbu, M. (2013). Stochastic perrons method for hamiltonjacobibellman equations. *SIAM J. Control Optim.*, 51(6):4274 – 4294.

[Crandall et al., 1992] Crandall, M. G., Ishii, H., and Lions, P. L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67.

[Fleming and Soner, 2006] Fleming, W. H. and Soner, H. M. (2006). *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition.

[Topp, 2014] Topp, E. (2014). Existence and uniqueness for integro-differential equations with dominating drift terms. *Comm. Partial Differential Equations*, 39(8):1523–1554.