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The Power Allocation Game on A Dynamic Network: Equilibrium Selection

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Abstract— **This note proposes two equilibrium selection methods and applies them to the power allocation game developed in [1]. The first method has the game take place on a sequence of time-varying graphs, which redefines the PAG in an extensive form game framework, and selects the subgame perfect Nash equilibria. The second method has the power allocation game take place on a different sequence of time-varying graphs and selects the "resilient" Nash equilibria, where the concept of "resilience" is taken from the literature of network security. Certain technical results as well as the link between the two methods will be discussed. Either method is also applicable to equilibrium selection problems involving other network games.**

Index Terms— **subgame perfect equilibrium, resilience, equilibrium selection, time-varying graph, extensive form, pure strategy Nash equilibrium**

I. Introduction

A power allocation game or PAG has been formulated in [1] as a distributed resource allocation game between n countries operating in a networked environment. The game is formulated on a simple, undirected, signed graph G whose n vertices correspond to the countries and whose m edges represent relationships between countries. In [1], pure strategy Nash equilibrium existence is established, and in [2], examples of equilibrium predictions of the PAG for countries survival are offered. The aim of this paper is to extend the formulation to time-varying graphs and to study how the extension can be used for selecting/refining pure strategy Nash equilibria of the PAG. In particular it will be shown that an equilibrium that is both subgame perfect and "resilient" exists in any variation of the extended game, and that subgame perfect or resilient equilibria with particular properties exist in certain types of the extended game. It appears that these basic ideas can also be useful for equilibrium selection problems in other network games, which will be discussed in an expanded version of this paper.

This paper is organized as follows. First the power allocation game formulated in [1] will be briefly summarized in Section II. Then in Section III the extended version will be described and finally in Section IV results appropriate to the generalization will be stated.

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II. The Power Allocation Game

A. Basic Idea

By the *power allocation game* or PAG is meant a distributed resource allocation game between n countries with labels in $\mathbf{n} = \{1, 2, \ldots, n\}$ [1]. The game is formulated on a simple, undirected, signed graph $\mathbb{G}$ called "an environment graph" [2] whose n vertices correspond to the countries and whose m edges represent relationships between countries. An edge between distinct vertices i and j, denoted by (i, j), is labeled with a plus sign if countries i an j are friends and with a minus sign if countries i and j are adversaries. Let the set of all friendly pairs be $\mathcal{R}_{\mathcal{F}}$ and the set of all adversarial pairs be $\mathcal{R}_{\mathcal{A}}$. For each $i \in \mathbf{n}$, $\mathcal{F}_i$ and $\mathcal{A}_i$ denote the sets of labels of country i's friends and adversaries respectively; it is assumed that $i \in \mathcal{F}_i$ and that $\mathcal{F}_i$ and $\mathcal{A}_i$ are disjoint sets. Each country i possesses a nonnegative quantity p_i called the *total power* of country i. An allocation of this power or *strategy* is a nonnegative $n \times 1$ row vector u_i whose j component u_{ij} is that part of p_i which country i allocates under the strategy to either support country j if $j \in \mathcal{F}_i$ or to demise country j if $j \in \mathcal{A}_i$; accordingly $u_ij = 0$ if $j \notin \mathcal{F}_i \cup \mathcal{A}_i$ and $u_{i1} + u_{i2} + \cdots + u_{in} = p_i$. The goal of the game is for each country to choose a strategy which contributes to the demise of all of its adversaries and to the support of all of its friends.

Each set of country strategies $\{u_i,\ i \in \mathbf{n}\}$ determines an $n \times n$ matrix U whose ith row is u_i. Thus $U = [u_{ij}]_{n\times n}$ is a nonnegative matrix such that, for each $i \in \mathbf{n}$, $u_{i1} + u_{i2} + \cdots + u_{in} = p_i$. Any such matrix is called a *strategy matrix* and $\mathcal{U}$ is the set of all $n \times n$ strategy matrices.

B. Multi-front Pursuit of Survival

In [1] and [2], how countries allocate the power in the support of *the survival* of its friends and the demise of that of its adversaries is studied, which is in line with the fundamental assumptions about countries' behavior in classical international relations theory. [**?**] The following additional formulations are offered:

Each strategy matrix U determines for each $i \in \mathbf{n}$, the *total support* $\sigma_i(U)$ of country i and the *total threat* $\tau_i(U)$ against country i. Here $\sigma_i : \mathcal{U} \to \mathbb{R}$ and $\tau_i : \mathcal{U} \to \mathbb{R}$ are non-negative valued maps defined by $U \longmapsto \sum_{j\in\mathcal{F}_i} u_{ji} + \sum_{j\in\mathcal{A}_i} u_{ij}$ and $U \longmapsto \sum_{j\in\mathcal{A}_i} u_{ji}$ respectively. Thus country i's total support is the sum of the amounts of power each of country i's friends allocate to its support plus the sum of the amounts of power country i allocates to the destruction of all of its adversaries. Country i's total threat, on the other hand, is the sum of the amounts of power country i's adversaries allocate to its destruction. These allocations in turn determine country i's *state* $x_i(U)$ which may be safe, precarious, or unsafe depending on the relative values of $\sigma_i(U)$ and $\tau_i(U)$. In particular, $x_i(U) =$ safe if $\sigma_i(U) > \tau_i(U)$, $x_i(U) =$ precarious if $\sigma_i(U) = \tau_i(U)$, or $x_i(U) =$ unsafe if $\sigma_i(U) < \tau_i(U)$.

In playing the PAG, countries select individual strategies in accordance with certain weak and/or strong preferences. A sufficient set of conditions for country i to *weakly prefer* strategy matrix $V \in \mathcal{U}$ over strategy matrix $U \in \mathcal{U}$ are as follows

1) For all $j \in \mathcal{F}_i$ either $x_j(V) \in \{\text{safe, precarious}\}$, or $x_j(U) \in \{\text{unsafe}\}$, or both.
2) For all $j \in \mathcal{A}_i$ either $x_j(V) \in \{\text{unsafe, precarious}\}$, or $x_j(U) \in \{\text{safe}\}$, or both.

Weak preference by country i of V over U is denoted by $U \preceq V$.

Meanwhile, a sufficient condition for country i to be *indifferent* to the choice between V and U is that $x_i(U) = x_j(V)$ for all $j \in \mathcal{F}_i \cup \mathcal{A}_i$. This is denoted by $V \sim U$.

Finally, a sufficient condition for country i to *strongly prefer* V over U is that $x_i(V)$ be a safe or precarious state and $x_i(U)$ be an unsafe state. Strong preference by country i of V over U is denoted by $U \prec V$.

III. PAG on Time-varying Graphs

Let $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ be called an "environment graph" as in [2]. In this paper, we will be interested in environment graphs which change over time. Let $\mathbb{G}(t)$ denote the environmental graph at time t for $t \in \{0, 1, 2, \ldots, n\}$. Let $\mathcal{F}_i(t)$ and $\mathcal{A}_i(t)$ respectively be the sets of labels of country i's friends and adversaries at time t. Two particular sequences of the environment graphs will be considered.

A. Sequence I: Subgame Perfect Nash Equilibrium

Let $\mathbb{G}$ be the fixed environment graph, and write $\mathcal{G}$ for the set of all spanning subgraphs of $\mathbb{G}$. A sequence of graphs $\mathbb{G}(t),\ t \in \{0, 1, 2, \ldots, n\}$ from $\mathcal{G}$ is an *ascending chain* if $\mathbb{G}(t) \subset \mathbb{G}(t+1),\ t \in \{0, 1, 2, \ldots, n\}$ where by $\mathbb{G}(t) \subset \mathbb{G}(t+1)$ we mean that the edge set of $\mathbb{G}(t)$ is contained in the edge set of $\mathbb{G}(t+1)$.

Sequence I: The ascending sequence $\mathbb{G}(t),\ t \in \{0, 1, 2, \ldots, n\}$ *reaches* $\mathbb{G}$ from $\mathbb{G}(0)$ in n steps, i.e., $\mathbb{G}(n) = \mathbb{G}$.

Rule I: At time $t = 0$, every country i simultaneously allocates power to its outgoing edges, namely its external friends and adversaries respectively in $\mathcal{F}_i - \{i\}$ and $\mathcal{A}_i$ in the environment of $\mathbb{G}(0)$; denote the power allocation matrix containing these allocations as $U(0)$. (For i, $u_{ii}(0)$ is uniquely determined once the allocations to i's outgoing edges are decided; therefore, the diagonal elements of U_0 can be suppressed).

At time $t \in \{1, 2, \ldots, n\}$ ($t \in \mathbb{Z}$, every country i keeps constant its allocations to its external friends and adversaries in $\{\mathcal{F}_i - \{i\}\}(t-1)$ and $\mathcal{A}_i(t-1)$, and allocates its reserved power $u_{ii}(t-1)$ to its new friends and adversaries respectively in $\mathcal{F}_i(t) - \mathcal{F}_i(t-1)$ and $\mathcal{A}_i(t) - \mathcal{A}_i(t-1)$.

For time $t \in \{1, 2, \ldots, n\}$, let the set of power allocation matrices (having suppressed the diagonal elements) at layer t be represented as $\mathcal{U}(t) \subset \mathbb{R}^{n \times n}$ where $\forall U(t) \in \mathcal{U}(t)$ and $i \in \mathbf{n}$, the total power constraint holds for country i, $\sum_{j \in \mathbf{n}} u_{ij}(t) = p_i$.

The information structure of the dynamic game is complete information. When making each possible allocation at time t, each country has observed *the power allocation path* prior to time t, which is

$$U(0), U(1), \ldots, U(t-1).$$

At the end of the sequence, each country i receives its state $x_i(U(t),\ t \in \{0, 1, 2, \ldots, n\})$ as the outcome of *the power allocation path* from $t = 0$ to $t = n$,

$$U(0), U(1), \ldots, U(n).$$

In other words, the power allocation outcome is only realized at $t = n$.

This dynamic game is the PAG (which is an infinite normal-form game) in extensive form. The first investigations of games in extensive form include [3], [4] and [5]; one famous application is the Stackelberg game that models market competition, where a leader acts first before the followers choose to whether to compete with it. Formally, the power allocation game in extensive form is a structure with a decision tree $\mathbb{T}$ with n layers and a nonempty set of decision nodes at layer t where $t \in \{0, 1, 2, \ldots, n\}$. Each decision node at each layer denotes the point the countries have to decide on the allocations on the friend and adversary relations just newly activated at t.

From each node at layer t, there grows an infinite number of branches, the qth of which represents a possible allocation strategy $U_q(t)$ made by countries to those new friends and adversaries. The number of the branches between any node at layer t and its successors at layer $t+1$ is the cardinality of $\mathcal{U}(t)$.

An immediate consequence of this extensive form game formulation is a set of subgames of countries' power allocation being derived. Before introducing the definition of subgames, the concept of information set is necessary. Each decision node in the tree T represents an "information set. As is commonly defined, an information set is a set of decision nodes that establishes all the possible allocations that could have taken place in the game so far, given what the players that will act next have observed. Assuming complete and perfect information (i.e., the power allocation path leading to the particular decision node has already been observed by countries), each information set in the tree is a singleton.

Usually, a subgame is defined as a game satisfying three criteria: first, it begins with a singleton information set; second, all successors of the initial decision node are contained in the subgame; third, if a decision node in a particular information set is in the subgame then all members of that information set belong to the subgame. Therefore, in this extensive form game framework, the q-th ($q \in \mathbb{N}$) decision node at layer t of T ($t \in \{0, 1, 2, \ldots, n\}$) and all its successors make up *a subgame at layer* t; let the set of subgames at layer t be $\kappa(t)$. Obviously, the total number of decision nodes in T equals the total number of subgames. Each path in the tree T represents a power allocation path from $t = 0$ to $t = n$, $U(0), U(1), \ldots, U(n)$. A function η :

$$\mathcal{U}(0) \times \mathcal{U}(1) \ldots \times \mathcal{U}(n) \longrightarrow \kappa(0) \times \kappa(1) \ldots \times \kappa(n)$$

maps a power allocation path to a sequence of $n+1$ subgames it has traversed, where the t-th subgame of this sequence can be represented as $\eta(U(0), U(1), U(n))_t, t \in \{0, 1, 2, \ldots, n\}$.

In the PAG in extensive form, it is natural to investigate the subgame perfect Nash equilibrium:

Definition 1 (Subgame Perfection Nash Equilibrium): A power allocation path

$$U^*(0), \ldots U^*(t) \ldots U^*(n)$$

based on Rule I is a subgame perfect Nash equilibrium for the PAG Γ in extensive form assuming an ascending sequence of environment graph if and only if there is no profitable one-shot deviations for any $i \in \mathbf{n}$.

Even though the PAG in extensive form is graphically represented by a tree of infinite branches, there is only a finite number of possible power allocation outcomes realized at the terminal nodes of the tree, i.e., a number of 3^n possible state vectors which countries will partially order based on the axioms in the setup of the PAG. Moreover, by definition, a power allocation path from $t = 0$ to $t = n$ is subgame perfect Nash equilibrium if and only if it is an equilibrium in all of the $n + 1$ subgames it traverses.

B. Sequence II: "Resilient" Nash Equilibrium

In the literature of computer networking, resilience is the ability to provide and maintain an acceptable level of service in the face of faults and challenges to the network. In the context of the power allocation game, intuitively speaking, if the environment graph suffers from certain "faults", such as a disappearance of a friend relation, and the allocations on the remaining graph still remain a pure strategy Nash equilibrium, it can be said that the original pure strategy Nash equilibrium is resilient to the change.

We say that the sequence of environmental graphs $\mathbb{G}(t),\ t \in \{0, 1, 2, \ldots, n\}$ from is a *descending chain* if $\mathbb{G}(t+1) \subset \mathbb{G}(t),\ t \in \{0, 1, 2, \ldots, n\}$ where by $\mathbb{G}(t+1) \subset \mathbb{G}(t+1)$ we mean that the edge set of $\mathbb{G}(t+1)$ is contained in the edge set of $\mathbb{G}(t)$.

Sequence II: Let the descending sequence $\mathbb{G}(t),\ t \in \{0, 1, 2, \ldots, n\}$ *starts at* $\mathbb{G}$ from $\mathbb{G}(0)$ and goes on for n steps, i.e., $\mathbb{G}(0) = \mathbb{G}$.

Rule II: At time $t \in \{0, 1, 2, \ldots, n\}$, every country i keeps its allocations to its external friends and adversaries respectively in $\{\mathcal{F}_i - \{i\}\}(t)$ and $\mathcal{A}_i(t)$ the same as it did toward them at $t - 1$, and updates its reserved power according to the following:

$$u_{ii}(t) = p_i - \sum_{j \in \{\mathcal{F}_i(t) - \{i\}\} \cup \mathcal{A}_i(t)} u_{ij}.$$

In other words, country i collects its prior power allocations on those friend and adversary relations that have disappeared at t back into its reserved power.

Denoting the PAG assuming the environment graph $\mathbb{G}(t)$ as $\Gamma(\mathbb{G}(t))$, a definition of *resilient* pure strategy Nash equilibrium is then:

Definition 2 (Resilient Nash Equilibrium): A power allocation matrix U is a resilient Nash Equilibrium for the PAG Γ if and only if, given a descending sequence of environment graphs satisfying Sequence II, the matrix $U(t)$ updated based on Rule II is a pure strategy Nash equilibrium in $\Gamma(\mathbb{G}(t))$.

IV. Results

Next we compare the equilibrium predictions of the PAG in extensive form to that of the original PAG in normal form.

Lemma 1: For any subgame prefect Nash equilibrium, there always exists an equivalent resilient Nash equilibrium. In other words, there can always be constructed a sequence of the environment graphs satisfying Sequence II, where a resilient Nash equilibrium exists, i.e., it is an equilibrium in any PAG assuming any environment graph on this sequence.

Lemma 2: Given a pure strategy Nash equilibrium U^*, if there exists two countries i and j such that $u^*_{ij} = u^*_{ji} = 0$, U^* is both subgame perfect and resilient Nash equilibrium.

Theorem 1: A power allocation matrix U^* of the PAG constructed using the algorithm in [1] is always subgame prefect and resilient Nash equilibrium.

Theorem 2: A power allocation matrix U^* of the PAG Γ

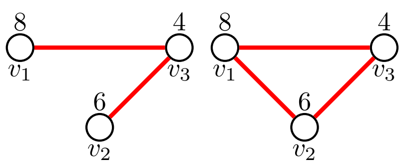
that is a *Balancing equilibrium* is subgame prefect and also resilient Nash equilibrium.

Theorem 3: In the PAG assuming an environment where no country has external friends and the adversary relations make up a complete graph, given *any nontrivial ascending chain* of environment graphs (a *trivial ascending chain* is where any subgraph in the chain is either empty or the environment graph $\mathbb{G}$ itself), *all* countries survive. (In [2], a country is said to survive if its total support exceeds or is equal to its total threats, or its state is either safe or precarious, that is, $\sigma_i(U) \geq \tau_i(U)$ or equivalently $x_i(U) =$ safe or precarious) in any subgame perfect Nash equilibrium if and only if there does not exist a country i such that $p_i > \sum_{j \in \mathcal{A}_i} p_j$.

Corollary 1: In the PAG assuming any environment, if there exists a country i such that $p_i > \sum_{j \in \mathcal{A}_i} p_j + \sum_{k \in \bigcup_{j \in \mathcal{A}_i} \mathcal{F}_j} p_k$ where $\bigcup_{j \in \mathcal{A}_i} \mathcal{F}_j$ denotes the set of friends of country i's adversaries, there exists subgame prefect equilibria (also resilient) in which a country j ($j \in \mathcal{A}_i$) is unsafe.

V. Example

An example of Theorem 3 is as follows. There are two environment graphs in the sequence illustrated in Figure 1. For better illustration, the vertices are labeled from v_1 to v_n.



(a) $\mathbb{G}_E(0)$ (b) $\mathbb{G}_E(1)$

Fig. 1: A sequence of environment graphs

The parameters of the PAG Γ are:

1) Set of countries' labels: $\mathbf{n} = \{1, 2, 3\}$
2) Countries' power: $[8 \quad 6 \quad 4]$.
3) Relations: $\mathcal{A}_1 = \{2, 3\}$, $\mathcal{A}_2 = \{1, 3\}$ and $\mathcal{A}_3 = \{1, 2\}$.
4) Preferences: assume the two preference axioms in [2].

No equilibrium in which only a country survives is subgame perfect Nash equilibrium in the PAG in extensive form assuming the graph sequence in Figure 1. The subgame perfect equilibrium classes are represented by the following state vectors, $[\text{safe}, \text{precarious}, \text{precarious}]$, $[\text{precarious}, \text{safe}, \text{precarious}]$, $[\text{precarious}, \text{precarious}, \text{safe}]$ and $[\text{precarious}, \text{precarious}, \text{precarious}]$. To see why this is so, when country 1 and country 2 get to decide on the allocations between them at $t = 1$ (because their adversary relation only appears at a later stage), the country with more remaining power may overwhelm the other country, making it unsafe. Then having anticipated it to be unsafe at $t = 1$, country i may deviate from its allocation strategy at $t = 0$ so as not to be the one with fewer remaining power. By this logic, any power allocation path leading to either country 1 or country 2 being unsafe at $t = 1$ is not an equilibrium. Two further take-away points are:

1) All countries will survive in any subgame prefect Nash equilibria of this game, and *this is independent of any nontrivial ascending chain of the environment graphs*.
2) However, it does depend on the sequence of the environment graphs that which country has to be precarious in all of the equilibria (in this example, country 3).

VI. Conclusion

A fully expanded paper based on this note will actually be the first part of a dynamic framework of the PAG in changing networked environments. This first part focuses specifically on the problem of equilibrium selection, whose formalization has a state vector realized only at the end of the power allocation path. As discussed, the proposed methods may be of independent interest, that is, being extrapolated to equilibrium selection problems in other network games. The second part will be on the problem of *countries' distributed, optimal control of their power allocation paths*, whose formulation will have a state or payoff vector realized at the end of every period on the power allocation path. In addition to countries' relations that may change over time, their power may also change based on a certain growth function. These aspects would make this second part not a simple generalization of the first part but address a completely different research question.

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