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Controlled Invariance for 1D and 2D Time-varying Nonlinear Control Systems

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***Abstract*— We study invariance properties with polynomially nonlinear and time-varying ODE systems (also termed one-dimensional or 1D systems due to the presence of one independent variable). The results are used to characterize time-varying controlled invariant varieties, that is, varieties that can be rendered invariant by state feedback. Analogous questions are also considered for a class of PDE systems in two independent variables (so-called two-dimensional or 2D systems), namely those which can be described by continuous Roesser models.**

I. INTRODUCTION

Controlled and conditioned invariant subspaces were introduced for time-invariant linear systems by Basile and Marro in 1969 and by Wonham and Morse in 1970 – see [1], [2] for comprehensive surveys. This geometric approach to control theory can be used to solve certain decoupling and noninteracting problems. The theory has been generalized to time-varying linear systems by Ilchmann [3] and to nonlinear systems by Isidori [4] and several other authors. Recently, some progress has been made in the area of polynomial systems [5], [6], [7], [8], [9], [10], where methods from symbolic computation can be used to test the conditions for controlled and conditioned invariance of varieties constructively. In the preprint [11], the polynomially nonlinear and time-varying case has been addressed in the general context of semi-algebraic sets (rather than varieties) and dynamic compensators (rather than static state feedback). Sections II and III of the present paper are based on this preprint, and the main contribution of the present manuscript is a partial generalization of results from [11] to systems given by certain PDE in two independent variables, namely the so-called Roesser models [12], [13].

Let K denote the field of real numbers or the field of rational numbers, and let $R := K[t, x_1, \dots, x_n]$ denote the polynomial ring in $1 + n$ variables over K. Consider the ordinary differential equation

$$\dot{x}(t) = F(t, x(t)), \tag{1}$$

where $F \in R^n$. Let $\varphi(t, t_0, x_0)$ denote the solution of the initial value problem

$$\dot{x}(t) = F(t, x(t)), \quad x(t_0) = x_0 \tag{2}$$

at time $t \in I(t_0, x_0)$, where $I(t_0, x_0)$ denotes the maximal existence interval of (2). By a time-varying set, we mean a set $S \subseteq \mathbb{R}^{1+n}$ and we write $S(t) := \{\xi \in \mathbb{R}^n \mid (t, \xi) \in S\}$. We will mostly be interested in time-varying varieties $V \subseteq \mathbb{R}^{1\times n}$. For this, let $p_1, \dots, p_k \in R$ be given and consider

$$V := \{(\tau, \xi) \mid p_i(\tau, \xi) = 0 \text{ for } 1 \le i \le k\}$$

and

$$V(t) := \{\xi \in \mathbb{R}^n \mid p_i(t, \xi) = 0 \text{ for } 1 \le i \le k\}.$$

We say that S is invariant for (1) if $x_0 \in S(t_0)$ implies that $\varphi(t, t_0, x_0) \in S(t)$ holds for all $t \in I(t_0, x_0)$.

In this paper, we derive a constructive test to decide whether a time-varying variety V is invariant for (1). This will be used to study the controlled invariance of V for the time-varying nonlinear control system

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t), \tag{3}$$

where $f \in R^n$ and $g \in R^{n\times m}$. One says that V is controlled invariant for (3) if there exists $\alpha \in R^m$ such that the feedback law $u(t) = \alpha(t, x(t))$ will lead to a closed loop system $\dot{x}(t) = F(t, x(t))$ with $F := f + g\alpha$ for which V is invariant.

The total time derivative of $p \in R$ along $F \in R^n$ is defined by

$$L_F(p) := \frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} F_i.$$

II. INVARIANCE

Lemma 1: Let $S \subseteq \mathbb{R}^{1+n}$ be a time-varying set. Define

$$\mathcal{J}(S) := \{p \in R \mid p(\tau, \xi) = 0 \text{ for all } (\tau, \xi) \in S\}.$$

If S is invariant for (1), then

$$L_F(p) \in \mathcal{J}(S) \text{ for all } p \in \mathcal{J}(S).$$

Proof: Consider (2) with $x_0 \in S(t_0)$ and let $\varphi(t, t_0, x_0)$ denote its solution at time t. By assumption, we have $\varphi(t, t_0, x_0) \in S(t)$ for all $t \in I(t_0, x_0)$. Let $p \in \mathcal{J}(S)$ be given. Then $p(t, \varphi(t, t_0, x_0)) = 0$ holds for all $t \in I(t_0, x_0)$. Taking the total time derivative, we get

$$L_F(p)(t, \varphi(t, t_0, x_0)) = 0.$$

Plugging in $t = t_0$, this yields

$$L_F(p)(t_0, x_0) = 0.$$

Since $(t_0, x_0) \in S$ was arbitrary, we may conclude that $L_F(p) \in \mathcal{J}(S)$. □

Theorem 1: Let $V \subseteq \mathbb{R}^{1+n}$ be a time-varying variety. Then V is invariant for (1) if and only if

$$L_F(p) \in \mathcal{J}(V) \text{ for all } p \in \mathcal{J}(V).$$

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Proof: Only the "if" part needs to be proven. Since $\mathcal{J}(V)$ is an ideal in the Noetherian ring R, it is finitely generated, say $\mathcal{J}(V) = \langle p_1, \ldots, p_k \rangle$. By assumption, we have

$$L_F(p_i) = \sum_{j=1}^{k} a_{ij} p_j$$

for some $a_{ij} \in R$. Let t_0, x_0 with $x_0 \in V(t_0)$ be given. Consider $y_i(t) := p_i(t, \varphi(t, t_0, x_0))$. Taking the total time derivative, we obtain

$$\dot{y}_i(t) = L_F(p_i)(t, \varphi(t, t_0, x_0)) = \sum_{j=1}^{k} a_{ij}(t, \varphi(t, t_0, x_0)) y_j(t).$$

Set $A_{ij}(t) := a_{ij}(t, \varphi(t, t_0, x_0))$ and note that A is continuous on $I(t_0, x_0)$. Thus $\dot{y}(t) = A(t)y(t)$ and $y(t_0) = 0$ imply that $y \equiv 0$ on $I(t_0, x_0)$. This means that $(t, \varphi(t, t_0, x_0)) \in V$, or equivalently, $\varphi(t, t_0, x_0) \in V(t)$ for all $t \in I(t_0, x_0)$. □

Corollary 1: Let $V \subseteq \mathbb{R}^{1+n}$ be a time-varying variety. Then V is invariant for (1) if and only if it is positive invariant for (1).

Proof: Only the "if" part needs to be proven. Using the notation introduced in the proof of Lemma 1, the function $t \mapsto p(t, \varphi(t, t_0, x_0))$ is smooth. Hence its vanishing for all $t_0 \leq t \in I(t_0, x_0)$ implies that its derivative at $t = t_0$ must be zero, that is, $L_F(p)(t_0, x_0) = 0$. This shows that positive invariance of V is sufficient to guarantee that $L_F(p) \in \mathcal{J}(V)$ for all $p \in \mathcal{J}(V)$. By Theorem 1, the latter condition already implies invariance of V. □

We remark that in our setting, the function $t \mapsto p(t, \varphi(t, t_0, x_0))$ is even real-analytic. Thus its vanishing for all $t_0 \leq t \in I(t_0, x_0)$ implies that it is identically zero on $I(t_0, x_0)$. This yields an alternative proof of Corollary 1.

The condition of Theorem 1 can be turned into a constructive test, since it is equivalent to

$$L_F(p_i) \in \langle p_1, \ldots, p_k \rangle \text{ for all } 1 \leq i \leq k,$$

where $\mathcal{J}(V) = \langle p_1, \ldots, p_k \rangle$. This is due to the fact that $L_F(\cdot)$ is additive and satisfies the product rule $L_F(ap) = L_F(a)p + aL_F(p)$. Let $p := [p_1, \ldots, p_k]^T$ and let

$$\frac{\partial p}{\partial x} = \begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \cdots & \frac{\partial p_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial p_k}{\partial x_1} & \cdots & \frac{\partial p_k}{\partial x_n} \end{bmatrix}$$

denote the Jacobian of p w.r.t. x. Moreover, let $\frac{\partial p}{\partial t} = [\frac{\partial p_1}{\partial t}, \ldots, \frac{\partial p_k}{\partial t}]^T$ and $L_F(p) = [L_F(p_1), \ldots, L_F(p_k)]^T$. Then V is invariant for (1) if and only if the inhomogeneous linear system of equations

$$[p_1 I_k, \ldots, p_k I_k]\gamma = L_F(p)$$

has a solution $\gamma \in R^{k^2}$. This condition can be tested using Gröbner basis algorithms [14], [15]. Moreover, given V, we may compute the set

$$M := \{F \in R^n \mid V \text{ is invariant for } \dot{x}(t) = F(t, x(t))\} \quad (4)$$

as follows: Solve the inhomogeneous linear system of equations

$$\begin{bmatrix} -\frac{\partial p}{\partial x} & p_1 I_k & \ldots & p_k I_k \end{bmatrix} \Gamma = \frac{\partial p}{\partial t} \quad (5)$$

over R. Then

$$M = \{\pi_n(\Gamma) \mid \Gamma \text{ solves } (5)\},$$

where π_n denotes the projection onto the first n components. We note that if M is nonempty, then it has the structure of an affine module, that is $M = \mu + M_0$, where μ is a particular element of M and $M_0 \subseteq R^n$ is a submodule.

Example: Let V be defined by the equation $t(x^2+y^2) = 1$, where we write x, y instead of x_1, x_2 for simplicity. Using the computer algebra system SINGULAR [16], it can be checked that $\mathcal{J}(V)$ is indeed generated by $t(x^2 + y^2) - 1$. Note that $V(t)$ is a circle of radius $\frac{1}{\sqrt{t}}$ for $t > 0$, and empty for $t \leq 0$. Thus the question of invariance is relevant only for positive times. Using SINGULAR, we obtain

$$M = -\frac{1}{2} \begin{bmatrix} x(x^2 + y^2) \\ y(x^2 + y^2) \end{bmatrix} + M_0,$$

where

$$M_0 = \langle \begin{bmatrix} -y \\ x \end{bmatrix}, \begin{bmatrix} txy \\ ty^2 - 1 \end{bmatrix}, \begin{bmatrix} tx^2 - 1 \\ txy \end{bmatrix} \rangle.$$

If we perform the same computation over $\mathbb{R}(t)[x, y]$, we get

$$M = -\frac{1}{2} \begin{bmatrix} x/t \\ y/t \end{bmatrix} + M_0.$$

III. CONTROLLED INVARIANCE

Consider the control system (3). A time-varying variety V is controlled invariant for (3) if there exists $\alpha \in R^m$ such that V is invariant for $\dot{x}(t) = F(t, x(t))$ with $F := f + g\alpha$. This is true if and only if $f + g\alpha$ belongs to the set $M = \mu + M_0$ defined in (4). However, $f + g\alpha \in \mu + M_0$ is equivalent to

$$f - \mu \in M_0 + \mathrm{im}(g).$$

This condition is another inhomogeneous linear system of equations over R. For this, let M_0 be generated by the columns of a matrix h. Then V is controlled invariant for (3) if and only if

$$[-g, h]\Delta = f - \mu \quad (6)$$

has a polynomial solution Δ. Moreover, the set of all feedback laws rendering V invariant is given by

$$A = \{\pi_m(\Delta) \mid \Delta \text{ solves } (6)\},$$

where π_m denotes the projection onto the first m components. The set A is either empty or it has the structure of an affine module.

IV. ROESSER MODELS

Roesser [12] proposed his (linear, discrete, and time-invariant) model for image processing in 1975. Since then, it has become one of the most popular models in two-dimensional systems theory. Its nonlinear, continuous, and time-varying version is given by

$$\begin{aligned} \frac{\partial x_1}{\partial t_1} &= F_1(t_1, x_1(t_1,t_2), t_2, x_2(t_1,t_2)) \\ \frac{\partial x_2}{\partial t_2} &= F_2(t_1, x_1(t_1,t_2), t_2, x_2(t_1,t_2)), \end{aligned} \tag{7}$$

where $R := K[t_1, x_{11}, \ldots, x_{1n_1}, t_2, x_{21}, \ldots, x_{2n_2}]$ and $F_i \in R^{n_i}$. Typically, t_i will be spatial variables, but we keep the allusion to time to emphasize the connection with the first part of the paper. Existence and uniqueness questions with these models have been studied by David et al. [17], [18]. Together with the initial data

$$x_1(t_1^0, t_2) = \psi_1(t_2) \quad \text{and} \quad x_2(t_1, t_2^0) = \psi_2(t_1),$$

where $\psi_i : \mathbb{R} \to \mathbb{R}^{n_i}$ are given smooth functions, the equations (7) admit a unique solution in some open neighborhood $I = I(t_1^0, \psi_1, t_2^0, \psi_2)$ of (t_1^0, t_2^0). Let $S_i \subseteq \mathbb{R}^{1+n_i}$ be time-varying sets and let $S_i(t_i) := \{\xi_i \in \mathbb{R}^{n_i} \mid (t_i, \xi_i) \in S_i\}$. Set $S := S_1 \times S_2 \subseteq \mathbb{R}^{1+n_1+1+n_2}$. We say that S is invariant for (7) if

$$\left.\begin{array}{l} \psi_1(t_2) \in S_1(t_1^0) \forall t_2 \\ \psi_2(t_1) \in S_2(t_2^0) \forall t_1 \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} x_1(t_1,t_2) \in S_1(t_1) \forall t \in I \\ x_2(t_1,t_2) \in S_2(t_2) \forall t \in I, \end{array}\right.$$

where $t = (t_1, t_2)$. In particular, let $V_i \subseteq \mathbb{R}^{1 \times n_i}$ be time-varying varieties, say, $V_1 = \{(\tau_1, \xi_1) \mid p_i(\tau_1, \xi_1) = 0\}$ and $V_2 = \{(\tau_2, \xi_2) \mid q_j(\tau_2, \xi_2) = 0\}$, where $p_i \in R_1 := K[t_1, x_{11}, \ldots, x_{1n_1}]$ and $q_j \in R_2 := K[t_2, x_{21}, \ldots, x_{2n_2}]$. Then

$$V = \{(\tau_1, \xi_1, \tau_2, \xi_2) \mid p_i(\tau_1, \xi_1) = 0, q_j(\tau_2, \xi_2) = 0\}.$$

The total time derivative w.r.t. t_1 of $p \in R_1$ along $F_1 \in R^{n_1}$ is given by

$$L_{F_1}(p) = \frac{\partial p}{\partial t_1} + \sum_{i=1}^{n_1} \frac{\partial p}{\partial x_{1i}} F_{1i}.$$

The total time derivative w.r.t. t_2 of $q \in R_2$ along $F_2 \in R^{n_2}$ is given by

$$L_{F_2}(q) = \frac{\partial q}{\partial t_2} + \sum_{j=1}^{n_2} \frac{\partial q}{\partial x_{2j}} F_{2j}.$$

Define

$$\mathcal{J}(S_1) = \{p \in R_1 \mid p(\tau_1, \xi_1) = 0 \text{ for all } (\tau_1, \xi_1) \in S_1\},$$

$$\mathcal{J}(S_2) = \{q \in R_2 \mid q(\tau_2, \xi_2) = 0 \text{ for all } (\tau_2, \xi_2) \in S_2\},$$

$$\text{and } \mathcal{J}(S) = \{r \in R \mid r \text{ vanishes on } S\}.$$

Lemma 2: If $S = S_1 \times S_2$ is invariant for (7), then

$$L_{F_1}(\mathcal{J}(S_1)) \subseteq \mathcal{J}(S) \quad \text{and} \quad L_{F_2}(\mathcal{J}(S_2)) \subseteq \mathcal{J}(S).$$

Proof: Let $(t_1^0, \xi_1, t_2^0, \xi_2) \in S$. Let $\psi_1 \equiv \xi_1$ and $\psi_2 \equiv \xi_2$ be constant functions. Then $\psi_1(t_2) = \xi_1 \in S_1(t_1^0)$ for all t_2 and $\psi_2(t_1) = \xi_2 \in S_2(t_2^0)$ for all t_1. Let $x = (x_1, x_2)$ denote the solution to (7) resulting from these initial data. Let $p \in \mathcal{J}(S_1)$. By assumption, $p(t_1, x_1(t_1, t_2)) = 0$ for all $(t_1, t_2) \in I$. Taking the total time derivative w.r.t. t_1, we obtain

$$L_{F_1}(p)(t_1, x_1(t_1,t_2), t_2, x_2(t_1,t_2)) = 0.$$

Plugging in $t_1 = t_1^0$, $t_2 = t_2^0$, this yields

$$L_{F_1}(p)(t_1^0, \xi_1, t_2^0, \xi_2) = 0.$$

Since $(t_1^0, \xi_1, t_2^0, \xi_2)$ was an arbitrary element of S, we may conclude that $L_{F_1}(p)$ vanishes on S. The second statement is analogous. □

Theorem 2: Let $V = V_1 \times V_2$ be a time-varying variety. Then V is invariant for (7) if and only if

$$L_{F_1}(\mathcal{J}(V_1)) \subseteq \mathcal{J}(V) \quad \text{and} \quad L_{F_2}(\mathcal{J}(V_2)) \subseteq \mathcal{J}(V).$$

Proof: Only the "if" part needs to be proven. We will use the fact that $\mathcal{J}(V_1 \times V_2) = \langle \mathcal{J}(V_1), \mathcal{J}(V_2) \rangle$ [13]. Let $\mathcal{J}(V_1) = \langle p_1, \ldots, p_k \rangle \subseteq R_1$ and $\mathcal{J}(V_2) = \langle q_1, \ldots, q_l \rangle \subseteq R_2$. Then $\mathcal{J}(V) = \langle p_1, \ldots, p_k, q_1, \ldots, q_l \rangle \subseteq R$. By assumption, we have

$$L_{F_1}(p_i) = \textstyle\sum_{j=1}^k a_{ij} p_j + \sum_{j=1}^l b_{ij} q_j$$

and

$$L_{F_2}(q_i) = \textstyle\sum_{j=1}^k c_{ij} p_j + \sum_{j=1}^l d_{ij} q_j$$

for some $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in R$. Let $\psi_1(t_2) \in V_1(t_1^0)$ for all t_2 and $\psi_2(t_1) \in V_2(t_2^0)$ for all t_1. Let $x = (x_1, x_2)$ denote the solution resulting from these initial data. Define $y_i(t_1, t_2) := p_i(t_1, x_1(t_1, t_2))$ and $z_i(t_1, t_2) := q_i(t_2, x_2(t_1, t_2))$. Taking the total time derivatives w.r.t. t_1 and t_2, respectively, we obtain

$$\begin{bmatrix} \frac{\partial y}{\partial t_1} \\ \frac{\partial z}{\partial t_2} \end{bmatrix} (t_1, t_2) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} (t_1, t_2) \begin{bmatrix} y \\ z \end{bmatrix} (t_1, t_2),$$

where $A_{ij}(t_1, t_2) := a_{ij}(t_1, x_1(t_1,t_2), t_2, x_2(t_1,t_2))$ and analogously for B, C, D. This is a linear time-varying Roesser model in which A, B, C, D are continuous functions of time. By assumption on the initial data, we have $y_i(t_1^0, t_2) = p_i(t_1^0, \psi_1(t_2)) = 0$ and $z_i(t_1, t_2^0) = q_i(t_2^0, \psi_2(t_1)) = 0$ for all i, that is, $y(t_1^0, t_2) = 0$ for all t_2 and $z(t_1, t_2^0) = 0$ for all t_1. By the existence and uniqueness results from [17], [18], this implies that y and z are identically zero on I. Thus $x_1(t_1, t_2) \in V_1(t_1)$ and $x_2(t_1, t_2) \in V_2(t_2)$ holds for all $(t_1, t_2) \in I$. □

Assume that $\mathcal{J}(V_1) = \langle p_1, \ldots, p_k \rangle$ and $\mathcal{J}(V_2) = \langle q_1, \ldots, q_l \rangle$. Set $p = [p_1, \ldots, p_k]^T$ and $q = [q_1, \ldots, q_l]^T$ and let $\frac{\partial p}{\partial x_1}$, $\frac{\partial q}{\partial x_2}$ denote their Jacobians w.r.t. x_i. Moreover, let $\frac{\partial p}{\partial t_1}$, $\frac{\partial q}{\partial t_2}$ and $L_{F_1}(p)$, $L_{F_2}(q)$ be defined as usual. Thus $V = V_1 \times V_2$ is invariant for (7) if and only if the inhomogeneous linear systems of equations

$$[p_1 I_k, \ldots, p_k I_k, q_1 I_k, \ldots, q_l I_k]\theta_1 = L_{F_1}(p)$$

and

$$[p_1 I_l, \ldots, p_k I_l, q_1 I_l, \ldots, q_l I_l]\theta_2 = L_{F_2}(q)$$

have solutions $\theta_1 \in R^{k(k+l)}$ and $\theta_2 \in R^{l(k+l)}$. Let M_i denote the set of all $F_i \in R^{n_i}$ such that V is invariant for (7). Consider

$$\left[\begin{array}{cccc} -\frac{\partial p}{\partial x_1} & p_1 I_k & \dots & q_l I_k \end{array} \right] \Theta_1 = \frac{\partial p}{\partial t_1} \qquad (8)$$

and

$$\left[\begin{array}{cccc} -\frac{\partial q}{\partial x_2} & p_1 I_l & \dots & q_l I_l \end{array} \right] \Theta_2 = \frac{\partial q}{\partial t_2}. \qquad (9)$$

Then

$$M_1 = \{\pi_{n_1}(\Theta_1) \mid \Theta_1 \text{ solves } (8)\}$$

and

$$M_2 = \{\pi_{n_2}(\Theta_2) \mid \Theta_2 \text{ solves } (9)\},$$

where π_{n_i} denotes the projection onto the first n_i components.

Example: Let $n_1 = 2$, $n_2 = 1$ and let $V_1 \subseteq \mathbb{R}^2$ be given by $t_1(x^2 + y^2) = 1$, $V_2 = \{0\}$. For simplicity, we denote the three partial states by x, y, z instead of x_{11}, x_{12}, x_{21}. Using SINGULAR, we may check that $\mathcal{J}(V_1)$ is generated by $t_1(x^2+y^2)-1$ and that $\mathcal{J}(V_2)$ is generated by z. Moreover, we can compute $M_2 = \langle z, t_1(x^2 + y^2) - 1\rangle$ and

$$M_1 = -\frac{1}{2} \left[\begin{array}{c} x(x^2 + y^2) \\ y(x^2 + y^2) \end{array} \right] + M_1^0,$$

where $M_1^0 =$

$$\langle \left[\begin{array}{c} z \\ 0 \end{array} \right], \left[\begin{array}{c} 0 \\ z \end{array} \right], \left[\begin{array}{c} -y \\ x \end{array} \right], \left[\begin{array}{c} t_1xy \\ t_1y^2 - 1 \end{array} \right], \left[\begin{array}{c} t_1x^2 - 1 \\ t_1xy \end{array} \right] \rangle.$$

Finally, consider the control system

$$\begin{array}{rcl} \frac{\partial x_1}{\partial t_1} & = & f_1(t_1, x_1, t_2, x_2) + g_1(t_1, x_1, t_2, x_2)u \\ \frac{\partial x_2}{\partial t_2} & = & f_2(t_1, x_1, t_2, x_2) + g_2(t_1, x_1, t_2, x_2)u, \end{array}$$

where x_1, x_2, and u depend on t_1, t_2. For controlled invariance, we need to check whether there exists $\alpha \in R^m$ such that

$$(f_1, f_2) + (g_1, g_2)\alpha \in M_1 \times M_2.$$

Since the sets M_i are affine modules, say, $M_i = \mu_i + M_i^0$ for some submodules $M_i^0 \subseteq R^{n_i}$, this amounts to another solvability test for an inhomogeneous linear system of equations over R. Let M_i^0 be generated by the columns of a matrix h_i. Then we need to test whether

$$\left[\begin{array}{ccc} -g_1 & h_1 & 0 \\ -g_2 & 0 & h_2 \end{array} \right] \Lambda = \left[\begin{array}{c} f_1 - \mu_1 \\ f_2 - \mu_2 \end{array} \right] \qquad (10)$$

has a polynomial solution Λ. Thus the set of all feedback laws rendering V invariant is given by

$$A = \{\pi_m(\Lambda) \mid \Lambda \text{ solves } (10)\},$$

which is either empty or an affine module.

CONCLUDING REMARK

In this paper, we have addressed 2D time-varying varieties of the special form $V(t_1, t_2) = V_1(t_1) \times V_2(t_2)$. The general case $V(t_1, t_2) = V_1(t_1, t_2) \times V_2(t_1, t_2)$ is a topic for future research. For instance: Given $\xi_1 \in V_1(t_1^0, t_2^0)$, is it always possible to find a smooth function ψ_1 of t_2 with $\psi_1(t_2^0) = \xi_1$ such that $\psi_1(t_2) \in V_1(t_1^0, t_2)$ for all $t_2 \in \mathbb{R}$? Such property is needed for generalizing Lemma 2 to the general situation.

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