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Distributed Resource Allocation Algorithm for Disturbed Multi-Agent Systems Over Weight-Balanced Digraphs

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***Abstract*—In this paper, a distributed resource allocation problem is investigated for disturbed continuous-time multi-agent systems, where the communication among agents is depicted by strongly connected and weight-balanced digraphs. A distributed continuous-time algorithm is developed based on internal model principle, and the algorithm can ensure the multi-agent systems exponentially converge to the optimal allocation even in the presence of external disturbance.**

***Index Terms*—Distributed optimization, resource allocation, multi-agent systems, disturbance rejection.**

I. Introduction

Owing to the fast development of large-scale systems/networks, the control and optimization of continuous-time multi-agent systems have drawn many research interests. In addition to multi-agent consensus [1], [2] and containment [3], distributed continuous-time optimization problems have attracted more and more attention such as [4] in recent years as general cooperative problems.

As one of the important optimization problems, resource allocation is a fundamental issue in many applications, such as power systems [5], and communication networks [6]. For example, [7] has developed a weighted gradient algorithm for a resource allocation problem with general objective functions.

For many physical systems, they inevitably suffer the influence of external disturbance. However, in most existing optimization tasks, the influence of external disturbance on system is not taken into account, such as [8]. The distributed optimization problem with external disturbance was studied in [9], in which reducing communication cost and subgradient measurement burden is not taken into account.

Many existing distributed algorithms for resource allocation problems rely on undirected graphs, such as [10]. It is well-known that balanced digraphs are less restrictive and more general than undirected graphs. There are a few results about the resource allocation with balanced directed communication networks (see [11]).

The motivation of this paper is to study a resource allocation problem of disturbed multi-agent systems. We propose a continuous-time algorithm to solve this problem and prove its convergence with the help of Lyapunov functions and convex analysis. The technical contributions are summarized as follows. (i) We consider a resource allocation problem of disturbed multi-agent systems over strongly connected and weight-balanced digraphs, which is an extension of the problems in [10], [11] by considering external disturbances and/or weight-balanced digraphs. (ii) We propose a distributed continuous-time internal model-based algorithm to solve the problem, where the internal model is used to reject external disturbances. (iii) We analyze the convergence of the algorithm, which can ensure the multi-agent systems exponentially converge to the optimal allocation.

The organization of this paper is as follows. In Section II, preliminaries are introduced and the considered resource allocation problem is formulated. In Section III, the main result is presented. Finally, in Section IV, conclusion is given.

Notations: $\mathbb{R}$ and $\mathbb{N}$ are the sets of real and natural numbers, respectively. $\mathbb{R}^n$ is the n-dimension Euclidean space. $\otimes$ and $\|\cdot\|$ denote the Kronecker product and the standard Euclidean norm, respectively. A^T is the transpose of matrix A. x_i is the ith element of vector x, and $col(x_1, ..., x_n) = [x_1^T, ..., x_n^T]^T$. I_n is a $n \times n$ identity matrix. 1_n and 0_n are the column vectors of n ones and zeros, respectively.

II. Preliminaries and Formulation

In this section, we first give some preliminary knowledge and then formulate our problem.

A. Graph Theory and Convex Analysis

The following concepts about graph theory can be found in [12]. Consider a network of N agents with interaction topology described by a directed graph (or simply a digraph) $\mathcal{G} := \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \dots, N\}$ is the node set, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. An edge of $\mathcal{G}$ is denoted by a pair of nodes $(i, j) \in \mathcal{E}$ if j can send its information to i. A path is a sequence of vertices connected by edges. A digraph is strongly connected if there is a path between any pair of vertices. A weighted digraph $\mathcal{G} := \{\mathcal{V}, \mathcal{E}, A\}$ consists of a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ and an adjacency matrix $\mathcal{A} = [a_{ij}]_{N\times N}$ with a_{ij} being the weighting of edge (i, j), where $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. It is noted that $a_{ii} = 0$ for any $i \in \mathcal{V}$, which indicates no self-connection in the graph. Besides, for an edge $(i, j) \in \mathcal{E}$, i is called the out-neighbor of j, and j is called the in-neighbor of i. The weighted in-degree and weighted out-degree of node i are $d_{in}^i = \sum_{j=1}^N a_{ij}$ and $d_{out}^i = \sum_{j=1}^N a_{ji}$, respectively. A digraph is weight-balanced if for any node $i \in \mathcal{V}$, the weighted in-degree and weighted out-degree coincide. The Laplacian matrix of $\mathcal{G}$ is $L = \mathcal{D}_{in} - A$, where $\mathcal{D}_{in} = diag\{d_{in}^1, \dots, d_{in}^N\} \in \mathbb{R}^{N\times N}$. Note that

$L1_N = 0$. Besides, $1_N^T L = 0$ and $L + L^T$ is positive semidefinite.

The eigenvalues of $L+L^T$ are denoted by $\hat{\lambda}_1, \dots, \hat{\lambda}_N$ with $\hat{\lambda}_i \leq \hat{\lambda}_j$ for $i \leq j$. For a strongly connected and weight-balanced digraph, zero is a simple eigenvalue of both L and $L + L^T$. The following definitions can be found in [13].

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y), \quad \forall\ x,\ y \in \mathbb{R}^n, \quad \forall\ \alpha \in [0,\ 1].$$

A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is ω-strong convex ($\omega > 0$) on $\mathbb{R}^n$ if

$$(x-y)^T(\nabla f(x) - \nabla f(y)) \geq \omega \|x-y\|^2,\ \forall\ x, y \in \mathbb{R}^n.$$

B. Problem Formulation

In the considered resource allocation problem, there are N agents, and their communication topology is described by a weight-balanced digraph $\mathcal{G}$ with the node set $\mathcal{V} = \{1, \cdots, N\}$ (where node i represents agent i). he dynamics of agent i is described by

$$\dot{x}_i = u_i + d_i, \quad i \in \mathcal{V}, \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^n$ are the state and control input of agent i, respectively, and d_i is the local disturbance generated by

$$\dot{\omega}_i = S\omega_i, \quad d_i = C\omega_i, \tag{2}$$

where $\omega_i \in \mathbb{R}^p$ and all the eigenvalues of $S \in \mathbb{R}^{p\times p}$ are distinct lying on the imaginary axis, which implies the boundedness of the disturbance.

Agent $i \in \mathcal{V}$ has a privately-known local cost function $f_i : \mathbb{R}^n \to \mathbb{R}$ and a local resource $d_i \in \mathbb{R}^n$. Besides, an allocation $x_i \in \mathbb{R}^n$ should be made by agent i. The objective for these N agents is to make suitable allocations such that minimize the global cost function $f(x)$ with $f(x) = \sum_{i=1}^N f(x_i)$. Moreover, the allocations made by these agents satisfy the network resource constraint, i.e., $\sum_{i=1}^N x_i = \sum_{i=1}^N d_i$, where $\sum_{i=1}^N d_i$ is the total network resource, consisted of all the local resources. To be strict, the resource allocation problem can be formulated as follows:

$$\begin{aligned} &\min_{x\in\mathbb{R}^{nN}} f(x), \quad f(x) = \sum_{i\in\mathcal{V}} f(x_i), \\ &\text{subject to } \sum_{i\in\mathcal{V}} x_i = \sum_{i\in\mathcal{V}} d_i. \end{aligned} \tag{3}$$

Our task is to provide a distributed resource allocation algorithm for every agent such that the decisions made by all agents not only satisfy the network resource constraint, but also minimize the global cost function.

In addition, the following assumption was widely used in the distributed resource allocation literature (e.g., [10], [7]).

Assumption 1. *The local cost function f_i is ω-strongly convex and differentiable with θ-Lipschitz gradient.*

Assumption 1 guarantees the existence and uniqueness of optimal solution to problem (3).

According to [14, Theorem 3.34], it is not hard to obtain the following optimality condition for problem (3).

Lemma 1. *If x_i^*, $(i \in \{1, \dots, N\})$ is the minimum of problem (3), then we have*

$$\nabla f_i(x_i^*) = \nabla f_j(x_j^*),\ \forall\, i, j \in \mathcal{V}, \quad \sum_{i=1}^N x_i^* = \sum_{i=1}^N d_i. \tag{4}$$

Conversely, if the condition (4) is satisfied for a feasible point x_i^* $(i \in \{1, \dots, N\})$ of (3), then x_i^* $(i \in \{1, \dots, N\})$ is the global minimum of problem (3).

III. Main Result

In this section, a distributed resource algorithms is first given in Subsection III-A. Then the convergence of the algorithm is analyzed in Subsection III-B.

A. Distributed Algorithm Design

A useful lemma about the local external disturbance is given first, whose proof can be found in [9].

Lemma 2. *Let $p(\lambda) = \lambda^s + p_1\lambda^{s-1} + \cdots + p_s$ be the minimal polynomial of S, and then the disturbance (2) can be rewritten as*

$$\dot{\tau}_i = (I_n \otimes \Phi)\tau_i, \quad d_i(t) = (I_n \otimes \Psi)\tau_i, \tag{5}$$

where $\tau_i = [\tau_{i1}^T \ \cdots \ \tau_{in}^T]^T$, $\tau_{ij} = [d_{ij}(t) \ \frac{dd_{ij}(t)}{dt} \ \cdots \ \frac{d^{s-1}d_{ij}(t)}{dt^{s-1}}]^T$, $j = 1, \dots, n$, $\Psi = \left[1 | 0_{1\times(s-1)}\right]$ *and*

$$\Phi = \left[\begin{array}{c|ccc} 0 & & I_{s-1} & \\ \hline -p_s & -p_{s-1} & \cdots & -p_1 \end{array}\right].$$

Clearly, there exists a vector ζ such that $F = \Phi + G\Psi$ is Hurwitz. As a result, there is a positive definite symmetric matrix P satisfying $F^T P + PF = -2I_s$.

For solving the resource allocation problem (3), the following distributed resource allocation algorithm is designed for agent i.

$$\begin{cases} \dot{x}_i = -\nabla f_i(x_i) - y_i - (I_n \otimes \Psi)\eta_i, \\ \dot{y}_i = k_1\left(\sum_{j=1}^N a_{ij}(z_i - z_j) - d_i + x_i\right) \\ \qquad - k_2 \sum_{j=1}^N a_{ij}(y_i - y_j), \\ \dot{z}_i = -\left(\sum_{j=1}^N a_{ij}(z_i - z_j) - d_i + x_i\right), \\ \dot{\eta}_i = (I_n \otimes F)\eta_i + (I_n \otimes G)u_i \end{cases} \tag{6}$$

To compensate the disturbances asymptotically, the term $\tau_i(t) - \eta_i(t)$ must vanish asymptotically. Performing a transformation $\bar{\eta}_i = \eta_i - \tau_i$ gives

$$\begin{aligned} \dot{\bar{\eta}}_i = &(I_n \otimes F)\bar{\eta}_i + (I_n \otimes G)(-\nabla f_i(x_i) \\ &- y_i - (I_n \otimes \Psi)\bar{\eta}_i), \end{aligned} \tag{7}$$

B. Convergence Analysis

Here we analyze the convergence of algorithm (6).

Let $x = col(x_1, \dots, x_N), y = col(y_1, \dots, y_N)$, $z = col(z_1, \dots, z_N), \overline{\eta} = col(\overline{\eta}_1, \dots, \overline{\eta}_N)$, $d = col(d_1, \dots, d_N)$. System (6) with the equation replaced by (7) can be rewritten in the following compact form:

$$\begin{cases} \dot{x} =- \nabla f(x) - y - (I_n \otimes \Psi)\overline{\eta}, \\ \dot{y} =k_1((L \otimes I_n)z - d + x) - k_2(L \otimes I_n)y, \\ \dot{z} =- ((L \otimes I_n)z - d + x), \\ \dot{\overline{\eta}} =(I_{Nn} \otimes F)\overline{\eta} + (I_{Nn} \otimes G)(-\nabla f(x) \\ \qquad - y - (I_n \otimes \Psi)\overline{\eta}). \end{cases} \tag{8}$$

Then, we study the property of the equilibrium point of algorithm (6), and then prove its convergence to the optimal solution.

Lemma 3. *Consider the resource allocation problem (3) over a strongly connected and weight-balanced digraph. If $(x^*, y^*, z^*, \overline{\eta}^*)$ is the equilibrium point of system (8), then x^* is the optimal solution of problem (3).*

Proof: Because $(x^*, y^*, z^*, \overline{\eta}^*)$ is the equilibrium point of system (8), we have

$$\begin{cases} - \nabla f(x^*) - y^* = 0, \\ k_1((L \otimes I_n)z^* - d + x^*) - k_2(L \otimes I_n)y^* = 0, \\ (L \otimes I_n)z^* - d + x^* = 0, \\ (I_{Nn} \otimes F)\overline{\eta}^* + (I_{Nn} \otimes G)(-\nabla f(x^*) \\ - y^* - (I_n \otimes \Psi)\overline{\eta}^*) = 0. \end{cases} \tag{9}$$

which implies that$(L \otimes I_n)y^* = 0_{Nn}$ and $(I_{Nn} \otimes F)\overline{\eta}^* = 0$, we can obtain that $\overline{\eta}^* = 0$.

Owing to $1_N^T L = 0$ and $\nabla f(x^*) = col(\nabla f_1(x_1^*), \dots, \nabla f_N(x_N^*))$, we have $\sum_{i=1}^N d_i = \sum_{i=1}^N x_i^*$ and $\nabla f_i(x_i^*) = \nabla f_j(x_j^*)$ for any $i, j \in \mathcal{V}$, which indicate that x^* is the optimal solution of problem (3) according to Lemma 3.1 of [15]. □

Next, we prove the convergence of system (8).

Theorem 1. *Under Assumption 1, consider the resource allocation problem (3) over a strongly connected and weight-balanced digraph. The algorithm (6) exponentially converges to the optimal solution of problem (3) if the local cost functions are differentiable with θ-Lipschitz gradients.*

Proof: We first define the following variables to obtain a standard stability problem:

$$\tilde{x} = x - x^*, \tilde{y} = y - y^*, \tilde{z} = z - z^*, \tilde{\eta} = \overline{\eta} - (I_{Nn} \otimes G)\tilde{x}. \tag{10}$$

For simplicity, let $n = 1$, and with the coordinate transformation (10), the following system is obtained via (8) and (9).

$$\begin{cases} \dot{\tilde{x}} =- (h + \tilde{y} + (I_{Nn} \otimes \Psi)(\tilde{\eta} + (I_{Nn} \otimes G)\tilde{x})), \\ \dot{\tilde{y}} =k_1(L\tilde{z} + \tilde{x}) - k_2 L\tilde{y}, \\ \dot{\tilde{z}} =- (L\tilde{z} + \tilde{x}), \\ \dot{\tilde{\eta}} =(I_{Nn} \otimes F)\tilde{\eta} + (I_{Nn} \otimes FG)\tilde{x}, \end{cases} \tag{11}$$

where $\tilde{h} = \nabla f(x + x^*) - \nabla f(x^*)$.

Obviously, after the coordinate transformation (10), the origin is the equilibrium of system (11). Thus, if $\tilde{x}$ tends to the origin, x converges to the optimal solution of problem (3). Herein, the next task is to prove the convergence of $\tilde{x}$.

For this purpose, we first perform the following transformation to simplify system (11)

$$\chi = (T \otimes I_n)\tilde{x}, \xi = (T \otimes I_n)\tilde{y}, \delta = (T \otimes I_n)\tilde{z} \tag{12}$$

where T is defined by $T^T = \begin{bmatrix} \frac{1}{\sqrt{N}} 1_N & R \end{bmatrix} = \begin{bmatrix} r & R \end{bmatrix}$

Based on the following orthogonal transformation,

$$\chi =col(\chi_1, \chi_2) = [r, R]^T \tilde{x}, \tag{13a}$$
$$\xi =col(\xi_1, \xi_2) = [r, R]^T \tilde{y}, \tag{13b}$$
$$\delta =col(\delta_1, \delta_2) = [r, R]^T \tilde{z}, \tag{13c}$$

Where $\chi_1, \xi_1, \vartheta_1 \in R^n$ and $\chi_2, \xi_2, \vartheta_2 \in R^{(N-1)n}$. Then from (11), we have

$$\begin{cases} \dot{\chi}_1 =- (r^T h + \xi_1 + (r^T \otimes I_n)\Phi), \\ \dot{\xi}_1 =k_1 \chi_1, \\ \dot{\delta}_1 =- \chi_1, \end{cases} \tag{14a}$$

$$\begin{cases} \dot{\chi}_2 =- (\xi_2 + R^T \tilde{h} + (R^T \otimes I_n)\Phi), \\ \dot{\xi}_2 =k_1(\chi_2 + R^T LR\delta_2) - k_2 R^T LR\xi_2, \\ \dot{\delta}_2 =- (\chi_2 + R^T LR\delta_2), \end{cases} \tag{14b}$$

With $\Phi = (I_{Nn} \otimes \Psi)(\tilde{\eta} + (I_{Nn} \otimes G)(T^{-1} \otimes I_n)\chi)$.

Consider the following candidate Lyapunov function

$$\begin{aligned} V_1 =& \frac{1}{2}\left(\frac{\omega + 1}{\omega}k_1 - 1\right)\|\chi_1\|^2 + \frac{1}{2}\|\chi_1 + \eta_1\| + \frac{1}{2\omega}\|\eta_1\|^2 \\ &+ \frac{\omega + 1}{2\omega}(k_1\|\chi_2\|^2 + \|\eta_2\|^2 + \|\delta_2\|^2), \end{aligned} \tag{15}$$

The derivative of V_1 along system (14) is

$$\begin{aligned} \dot{V}_1 =& - \frac{\omega + 1}{\omega}k_1 \chi_1^T r^T \tilde{h} - \frac{\omega + 1}{\omega}k_1 \chi_2^T R^T \tilde{h} - \|\eta_1\|^2 \\ &- \frac{\omega + 1}{\omega}k_2 \eta_2^T R^T LR\eta_2 - \frac{\omega + 1}{\omega}\delta_2^T R^T LR\delta_2 \\ &+ k_1\|\chi_1\|^2 - \eta_1^T r^T \tilde{h} - \frac{\omega + 1}{\omega}\delta_2^T \chi_2 \\ &+ \frac{\omega + 1}{\omega}k_1 \eta_2^T R^T LR\delta_2 - k_1 \chi_2^T (R^T \otimes I_n)\Phi \\ &- (\frac{\omega + 1}{\omega}k_1 \chi_1^T + \xi_1^T)(r^T \otimes I_n)\Phi. \end{aligned} \tag{16}$$

Since the digraph is strongly connected and weight-balanced,

$$\eta_2^T R^T LR\eta_2 \geq \frac{1}{2}\hat{\lambda}_2\|\eta_2\|^2, \tag{17a}$$
$$\delta_2^T R^T LR\delta_2 \geq \frac{1}{2}\hat{\lambda}_2\|\delta_2\|^2. \tag{17b}$$

Owing to the strong convexity of local cost functions and the orthogonal transformation (13), we obtain

$$-(\chi_1^T r^T \tilde{h} + \chi_2^T R^T \tilde{h}) \leq - \omega\|\chi\|^2. \tag{18}$$

Moreover, based on the inequality $ab \leq \frac{c}{2}a^2 + \frac{1}{2c}b^2$ for $c > 0$ and the Lipschitz property of the gradients of local cost functions, we have

$$-\delta_2^T \chi_2 \leq \frac{1}{8}\hat{\lambda}_2\|\delta_2\|^2 + \frac{2}{\hat{\lambda}_2}\|\chi_2\|^2, \tag{19a}$$

$$k_1\eta_2^T R^T L R\delta_2 \leq \frac{1}{8}\hat{\lambda}_2\|\delta_2\|^2 + 2\frac{\|L\|^2 k_1^2}{\hat{\lambda}_2}\|\eta_2\|^2. \tag{19b}$$

and

$$-\eta_1^T r^T \tilde{h} \leq \frac{1}{2}(\|\eta_1\|^2 + \theta^2\|\chi\|^2). \tag{20}$$

Also, because $\|\Phi\| \leq l_1\|\tilde{\eta}\| + l_2\|\chi\|$ for two positive numbers l_1, l_2, by completing the squares,

$$\begin{aligned}
&-(\frac{\omega+1}{\omega}k_1\chi_1^T + \xi_1^T)(r^T \otimes I_n)\Phi \leq (\frac{\omega+1}{2\omega}k_1^2 + \frac{1}{2}l_2^2 \\
&+\frac{\omega+1}{\omega}k_1 l_2)\|\chi\|^2 + (\frac{\omega+1}{2\omega}l_1^2 + \frac{1}{2} + l_1)\|\eta\|^2, \\
&-k_1\chi_2^T(R^T \otimes I_n)\Phi \leq (\frac{1}{2}k_1^2 + k_1 l_2)\|\chi\|^2 + \frac{1}{2}l_1^2\|\chi\|^2.
\end{aligned} \tag{21}$$

It results from (16), (17), (18), (19), (20) and (21) that

$$\begin{aligned}
\dot{V}_1 \leq &-\Bigg(\omega k_1 - \frac{1}{2}\theta^2 - \frac{2(\omega+1)}{\hat{\lambda}_2\omega} - \frac{1}{2}k_1^2 - k_1 l_2 \\
&- \frac{\omega+1}{2\omega} - \frac{\omega+1}{\omega}k_1 l_2 - \frac{1}{2}l_2^2\Bigg)\|\chi\|^2 \\
&- \frac{1}{2}\|\xi_1\|^2 - \frac{\hat{\lambda}_2(\omega+1)}{4\omega}\|\delta_2\|^2 \\
&- \frac{\hat{\lambda}_2(\omega+1)}{2\omega}\left(k_2 - \frac{4\|L\|^2 k_1^2}{\hat{\lambda}_2^2}\right)\|\xi_2\|^2 \\
&+ (\frac{1}{2}l_1 + \frac{\omega+1}{2\omega}l_1^2 + \frac{1}{2} + l_1)\|\eta\|^2.
\end{aligned}$$

Then, let us check the $\tilde{\eta}$ system. From lemma (2), we can take $V_0 = \tilde{\eta}^T(I_{Nn} \otimes P)\tilde{\eta}$ give $\dot{V}_0 \leq -\|\tilde{\eta}\|^2 + l_0\|\chi\|^2$ for a positive real number l_0.

Take the following Lyapunov function candidate for the whole system $V = V_1 + l_3 V_0, l_3 = \frac{1}{2}l_1 + \frac{\omega+1}{2\omega}l_1^2 + l_1 + 1$

Then, we have

$$\begin{aligned}
\dot{V} \leq &-\Bigg(\omega k_1 - \frac{1}{2}\theta^2 - \frac{2(\omega+1)}{\hat{\lambda}_2\omega} - \frac{1}{2}k_1^2 - k_1 l_2 \\
&- \frac{\omega+1}{2\omega} - \frac{\omega+1}{\omega}k_1 l_2 - \frac{1}{2}l_2^2 - l_3 l_0\Bigg)\|\chi\|^2 \\
&- \frac{\hat{\lambda}_2(\omega+1)}{4\omega}\|\delta_2\|^2 - \frac{1}{2}\|\tilde{\eta}\|^2 - \frac{1}{2}\|\xi_1\|^2 \\
&- \frac{\hat{\lambda}_2(\omega+1)}{2\omega}\left(k_2 - \frac{4\|L\|^2 k_1^2}{\hat{\lambda}_2^2}\right)\|\xi_2\|^2
\end{aligned}$$

Where $k_2 > \frac{4\|L\|^2 k_1^2}{\hat{\lambda}_2^2}$ and taking k_1 satisfying $-\big(\omega k_1 - \frac{1}{2}\theta^2 - \frac{2(\omega+1)}{\hat{\lambda}_2\omega} - \frac{1}{2}k_1^2 - k_1 l_2 - \frac{\omega+1}{2\omega} - \frac{\omega+1}{\omega}k_1 l_2 - \frac{1}{2}l_2^2 - l_3 l_0\big) \geq 1$. Then it follows from above that the system (14) exponentially converges to the origin, *i.e.*, x exponentially converges to x^* based on the previous analysis and Lemma 3. □

IV. Conclusions

In this paper, a continuous-time resource allocation problem with weight-balanced digraphs and external disturbance has been investigated. We considered problem degrades into the differentiable resource allocation problem, a simplified algorithm has been obtained, under which the allocated decision can exponentially converge to the exact optimal solution.

References

[1] W. F. Hu, L. Liu, and G. Feng, "Consensus of linear multi-agent systems by distributed event-triggered strategy," *IEEE Transactions on Cybernetics*, vol. 46, no. 1, pp. 148–157, Jan. 2016.

[2] S. Z. Su, Z. L. Lin, and A. Garcia, "Distributed synchronization control of multiagent systems with unknown nonlinearities," *IEEE Transactions on Cybernetics*, vol. 46, no. 1, pp. 325–338, Jan. 2016.

[3] Y. Lou and Y. Hong, "Target containment control of multi-agent systems with random switching interconnection topologies," *Automatica*, vol. 48, no. 5, pp. 879–885, May 2012.

[4] D. M. Yuan, D. W. C. Ho, and S. Y. Xu, "Regularized primal-dual subgradient method for distributed constrained optimization," *IEEE Transactions on Cybernetics*, vol. 46, no. 9, pp. 2109–2118, Sep. 2016.

[5] P. Yi, Y. Hong, and F. Liu, "Distributed gradient algorithm for constrained optimization with application to load sharing in power systems," *Systems & Control Letters*, vol. 83, pp. 45–52, Sep. 2015.

[6] D. P. Palomar and M. Chiang, "Alternative distributed algorithms for network utility maximization: Framework and applications," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2254–2269, Dec 2007.

[7] L. Xiao and S. Boyd, "Optimal scaling of a gradient method for distributed resource allocation," *Journal of Optimization Theory and Applications*, vol. 129, no. 3, pp. 469–488, Jun. 2006.

[8] Z. Deng, X. Wang, and Y. Hong, "Distributed optimisation design with triggers for disturbed continuous-time multi-agent systems," *IET Control Theory & Applications*, vol. 11, no. 2, pp. 282–290, Jan. 2017.

[9] X. Wang, D. Xu, and Y. Hong, "Consensus control of nonlinear leader-follower multi-agent systems with actuating disturbances," *Systems & Control Letters*, vol. 73, pp. 58–66, Nov. 2014.

[10] P. Yi, Y. Hong, and F. Liu, "Initialization-free distributed algorithms for optimal resource allocation with feasibility constraints and its application to economic dispatch of power systems," *Automatica*, vol. 74, pp. 259–269, Dec. 2016.

[11] A. Cherukuri and J. Cortés, "Distributed generator coordination for initialization and anytime optimization in economic dispatch," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 3, pp. 226–237, 2015.

[12] C. D. Godsil and G. Royle, *Algebraic Graph Theory*. New York: Springer, 2001.

[13] R. T. Rockafellar, *Convex Analysis*. Princeton, N.J.: Princeton University Press, 1970.

[14] A. P. Ruszczyński, *Nonlinear Optimization*. Princeton, N.J.: Princeton University Press, 2006.

[15] H. Lakshmanan and D. P. De Farias, "Decentralized resource allocation in dynamic networks of agents," *SIAM Journal on Optimization*, vol. 19, no. 2, pp. 911–940, 2008.