An Approximating Control Design for Optimal Mixing by Stokes Flows

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Consider a passive scalar field advected by an unsteady Stokes flow on an open bounded and connected domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, with a sufficiently smooth boundary Γ. The scalar field is governed by the transport equation, where molecular diffusion is assumed to be negligible and mixing is purely driven by advection. This naturally leads to the study of optimal mixing via an active control of the flow velocity. As discussed in our previous work [6], we consider the flow velocity induced by control inputs acting tangentially on the boundary of the domain through the Navier slip boundary conditions. This is motivated by the fact that moving walls accelerate mixing compared to fixed walls (cf. [2, 3, 4, 13, 15]). We aim at designing an optimal Navier slip boundary control that optimizes mixing at a given final time. The governing system of equations is

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \tag{0.1}$$

$$\frac{\partial v}{\partial t} - \Delta v + \nabla p = 0, \tag{0.2}$$

$$\nabla \cdot v = 0, \quad x \in \Omega, \tag{0.3}$$

with Navier slip boundary conditions (cf. [12]),

$$v \cdot n|_{\Gamma} = 0 \quad \text{and} \quad kv + (\mathbb{T}(v) \cdot n)_{\tau}|_{\Gamma} = g, \tag{0.4}$$

and the initial condition is given by

$$(\theta(0), v(0)) = (\theta_0, v_0), \tag{0.5}$$

where θ is the density, v is the velocity, p is the pressure, and g is the boundary control input, which is employed to generate the velocity field for mixing. Navier slip boundary conditions admit the fluid to slip with resistance on the boundary. Here n and τ denote the outward unit normal and tangentially vectors with respect to the domain Ω, $\mathbb{T}(v) = 2\mathbb{D}(v)$ with $\mathbb{D}(v) = (1/2)(\nabla v + (\nabla v)^T)$, $(\mathbb{T}(v) \cdot n)_{\tau}$ denotes the tangential component of $(\mathbb{T}(v) \cdot n)$, and $g \cdot n|_\Gamma = 0$. The friction between the fluid and the wall is proportional to $-v$ with the positive coefficient of proportionality k.

Due to the divergence-free and no-penetration boundary conditions imposed on the velocity field, it can be shown that any L^p-norm of θ is conserved (cf. [5, 6]), i.e.,

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad t \geq 0, \quad p \in [0, \infty]. \tag{0.6}$$

To qualify mixing, the mix-norm and negative Sobolev norms H^{-s}, for any $s > 0$, are usually adopted, especially for the scalar field with no molecular diffusion, based on ergodic theory (cf. [8, 9, 10, 11, 14]). The bridge that connects mixing with negative Sobolev norms is the property of weak convergence. As discussed in our previous work, we consider a general bounded domain for mixing and replace the negative Sobolev norm by the norm for the dual space $(H^s(\Omega))'$ of $H^s(\Omega)$ with $s > 0$. Also, we identify the space $(H^s(\Omega))', s > 0$ as the domain of operator Λ^{-s} equipped with the norm $\|\cdot\|_{(H^s(\Omega))'}$, where Λ is self-adjoint, positive and unbounded in $L^2(\Omega)$. Thus, $\Lambda^{2s} \in \mathcal{L}(H^s(\Omega), (H^s(\Omega))')$. In the rest of our discussion, we set $s = 1$.

Throughout this work, we use $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ for the L^2-inner products in the interior of the domain Ω and on the boundary Γ, respectively. The optimal control problem is formulated as follows: For a given $T > 0$, find a control g minimizing the cost functional

$$J(g) = \frac{1}{2}\|\theta(T)\|^2_{(H^1(\Omega))'} + \frac{1}{2}\|g\|^2_{U_{\mathrm{ad}}}, \tag{P}$$

subject to (0.1)–(0.5), where $\|\theta(T)\|_{(H^1(\Omega))'} = \|\Lambda^{-1}\theta(T)\|_{L^2(\Omega)}$ and U_{ad} is the set of admissible controls, which is often determined based on the physical properties as well as the need to establish the well-posedness of the problem, i.e., the existence of an optimal solution. In fact, the existence of an optimal solution to problem (P) can be proven for $U_{ad} = L^2(0, T; V_n^0(\Gamma))$, where $V_n^0(\Gamma) = \{g \in L^2(\Gamma) \colon \ g \cdot n|_\Gamma = 0\}$. The challenge arises in deriving the first-order necessary conditions of optimality. To establish the well-posedness of the Gâteaux derivative of θ, one needs $\sup_{t\in[0,T]} \|\nabla\theta\|_{L^2} < \infty$, which requires $\theta_0 \in H^1(\Omega)$ and the flow velocity to satisfy

$$\int_0^T \|\nabla v\|_{L^\infty(\Omega)}\, d < \infty.$$

Therefore, the initial condition v_0 and U_{ad} were chosen in a way such that this estimate holds [6]. To this purpose, the time regularity of g was needed. For computational convenience, the first derivative $\partial g/\partial t$ was adopted rather than the lower order fractional time derivative in the cost functional. Consequently, the optimality condition involved the time derivative of g, and thus the optimality system became difficult to further analyze the uniqueness of the solution.

0.1 An approximating control approach

In the current work, we start with investigating the approximating control problem by adding a small diffusion term $\epsilon\Delta\theta$, for $\epsilon > 0$, to the transport equation. The problem is now formulated as follows: For a given $T > 0$, find a control $g_\epsilon \in U_{\epsilon_{\text{ad}}} = L^2(0,T;V_n^0(\Gamma))$ minimizing the cost functional

$$J_\epsilon(g_\epsilon) = \frac{1}{2}\|\theta_\epsilon(T)\|^2_{(H^1(\Omega))'} + \frac{1}{2}\|g_\epsilon\|^2_{U_{\epsilon_{\text{ad}}}}, \qquad (\mathrm{P}_\epsilon)$$

subject to an approximating system governed by

$$\frac{\partial\theta_\epsilon}{\partial t} - \epsilon\Delta\theta_\epsilon + v_\epsilon\cdot\nabla\theta_\epsilon = 0, \tag{0.7}$$

$$\frac{\partial v_\epsilon}{\partial t} - \Delta v_\epsilon + \nabla p_\epsilon = 0, \tag{0.8}$$

$$\nabla\cdot v_\epsilon = 0, \quad x\in\Omega, \tag{0.9}$$

with the Neumann boundary condition for the scalar

$$\epsilon\frac{\partial\theta_\epsilon}{\partial n}|_\Gamma = 0 \tag{0.10}$$

and the nonhomogenous Navier slip boundary conditions for the velocity

$$v_\epsilon\cdot n|_\Gamma = 0 \quad \text{and} \quad (kv_\epsilon + (\mathbb{T}(v_\epsilon)\cdot n)_\tau)|_\Gamma = g_\epsilon. \tag{0.11}$$

The initial condition is given by

$$(\theta_\epsilon(0), v_\epsilon(0)) = (\theta_0, v_0). \tag{0.12}$$

Note that due to one-way coupling, the flow velocity v does not depend on ϵ and thus we have

$$v_\epsilon = v. \tag{0.13}$$

However, to distinguish the approximating system from the original one, we still adopt the notation v_ϵ.

The outline of the rest of this work is as follows. We first recall the basic results on Navier slip boundary control for Stokes problem. Then we establish the convergence of the approximating system governed by (0.7)–(0.12) to the original one governed by (0.1)–(0.5). Next we show the existence of an optimal solution to the approximating control problem (P_ϵ) and derive the first-order necessary conditions of optimality by using a variational inequality. Moreover, we prove that the optimal solution $(g^*_\epsilon, v^*_\epsilon, \theta^*_\epsilon)$ to problem (P_ϵ) strongly converges to (g^*, v^*, θ^*) as $\epsilon \to 0$, which turns out to be the optimal solution to the original problem (P). Finally, we prove that (g^*, v^*, θ^*) is unique for $d = 2$ and γ sufficiently large .

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