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Optimization Perspectives of Mean-Field Games

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Abstract— **This paper studies the connections between mean-field games and an auxiliary optimization problem, and compares the auxiliary optimization with potential functions. We formulate a large-population game in function space. The cost functions of all agents are weakly coupled though the mean of the population states/controls. We show that under some conditions, the ϵ-Nash equilibrium of the mean-field game is the optimal solution to an auxiliary optimization problem, and this is true even when the optimization problem is non-convex. The result enables us to evaluate the mean-field equilibrium, and also has some interesting implications on the existence, uniqueness and computation of the mean-field equilibrium. While the auxiliary optimization is similar to the potential function in potential games, we show that in general, the mean-field game considered in this paper is not a potential game. We compare the auxiliary optimization problem with potential function minimization, and discuss their differences in terms of solution concept and computation complexity.**

I. Introduction

Mean-field games study the interactions among a large population of agents. The decisions of these agents are coupled through a mean-field term that depends on the statistical information of the entire population. When the population is large, the impact of individual decision is small. We can then characterize the game equilibrium via the interactions between the agent and the mean field, instead of considering the detailed interactions among all the agent. This idea was originally proposed by Lasry and Lions [1], [2] and by Huang et al. [3], [4], where the game equilibrium is captured by a set of coupled backward Hamilton-Jacobi-Bellman equation and forward Fokker-Planck-Kolmogorov equation. These seminal results attracted numerous research attentions. They lead to extensive results on the existence [4], [5], uniqueness [1], [6], and computation [7], [8], [9] of the mean-field equilibrium. For a more comprehensive review, please refer to [10] and [11].

Different from the aforementioned literature, another strand of works focus on the efficiency of the mean-field equilibrium. In these papers, the connections between the mean-field game and the social welfare optimization is studied. Along this line, [12] and [13] showed that the coordinator can design a mean-field game to asymptotically achieve social optima as the population size goes to infinity. This result is true only when the game is *cooperative*. In non-cooperative setting, a recent work [14] showed that the Nash equilibrium of an electric vehicle charging game is socially optimal when the mean-field game is a potential game. In the case of non-cooperative and non-potential games, the mean-field equilibrium is shown to be inefficient in general. For instance, [15] employed a variational approach to study the efficiency loss of mean-field equilibria for a synchronization game among oscillators. In [16], a mean-field congestion game was formulated, and numerical results were presented to show that the mean-field equilibrium is inefficient in general. In [17], the authors showed that the mean-field equilibrium is the optimal solution to a *modified* social welfare optimization when the social welfare function is convex.

Similar to [17], this paper studies the mean-field games from the optimization perspective: we connect the mean-field equilibrium to an auxiliary optimization problem. A large-population game is formulated in vector spaces. Each agent seeks to minimize a cost function, and the cost couples with others through a mean field term that depends on the mean of the population states/controls. The key contributions of this paper are summarized as follows.

- First, we show that under some conditions, the ϵ-Nash equilibrium of the mean-field game is the solution to an auxiliary optimization problem. On the other hand, any solution to the auxiliary optimization problem is also an ϵ-Nash equilibrium of the game. Different from [17], this is true even when the optimization problem is non-convex. Our result establishes connections between the mean-field game and the auxiliary optimization problem. Using this connection, we can study the property of the mean-field equilibrium by looking at the optimal solution to the auxiliary optimization problem. Since the latter is well-studied, this connection can lead to new results on the existence, uniqueness and computation of the mean-field equilibrium.
- We compare the difference between the auxiliary optimization and the potential function minimization. We show that in general, the mean-field game considered in this paper is not a potential game, and the auxiliary optimization is not a potential function. On the other hand, under some additional assumptions (i.e., mean-field coupling term is linear), the mean-field game may reduce to a potential game. In this case, either the potential function or the auxiliary optimization can be used to derive the mean-field equilibrium. We show that compared to the potential function method, the proposed auxiliary optimization provides a relaxed Nash equilibrium, which enables decentralized implementation and

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enjoys better scalability.

The paper proceeds as follows. The mean-field game is formulated in Section II. The solution to the game problem and its connections to the auxiliary optimization is studied in Section III. Section IV compares the auxiliary optimization problem with the potential function minimization.

II. Problem Formulation

This section formulates the mean-field game in vector spaces. It includes both discrete-time and continuous-time problems as special cases, and addresses both deterministic and stochastic problems. The rest of this section presents the mathematical formulation of this problem.

A. The Mean-Field Game

Consider game in a vector space with N agents. Each agent i has a state variable x_i, a control input $u_i \in \mathcal{U}_i$ and a noise input π_i, where $\mathcal{U}_i$ is an arbitrary vector space, and π_i is a random element in a measurable space $(\Pi, \mathcal{B}_i)$ with an underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The state of each agent is determined by the control and noise according to the following mapping $f_i : \mathcal{U}_i \times \Pi_i \to \mathcal{X}$:

$$x_i = f_i(u_i, \pi_i), \quad u_i \in \mathcal{U}_i, \tag{1}$$

where x_i is a random element that takes value in the space $\mathcal{X}$. To ensure that x_i is well-defined, we impose the following assumption on $f_i(u_i, \pi_i)$:

Assumption 1: For each $u_i \in \mathcal{U}_i$, $f_i(u_i, \pi_i(\cdot)) : \Omega \to \mathcal{X}$ is a measurable mapping with respect to $\mathcal{F}/\mathcal{Z}$, where $\mathcal{Z}$ is a σ-algebra on $\mathcal{X}$,

Under Assumption 1, $x_i : \Omega \to \mathcal{X}$ is a measurable mapping with respect to $\mathcal{F}/\mathcal{Z}$. Therefore, x_i is a well-defined random element that takes value in $\mathcal{X}$. On the space $\mathcal{X}$, we define an inner product and a norm. In particular, denote the inner product as $x \cdot y$ for $x, y \in \mathcal{X}$, and define the norm as $||x|| = \sqrt{x \cdot x}$. We assume that $\mathcal{X}$ is complete.

Assumption 2: $\mathcal{X}$ is a Hilbert space.

Throughout the paper, we assume that x_i and x_j are independent. In addition, we assume that x_i has bounded second moment, i.e., there exists $\bar{C} \geq 0$ such that $\mathbb{E}||x_i||^2 \leq \bar{C}$ for all $i = 1, \ldots, N$. In this case, the admissible control set can be defined as $\bar{\mathcal{U}}_i = \{u_i \in \mathcal{U}_i | x_i = f_i(u_i, \pi_i), \mathbb{E}||x_i||^2 \leq \bar{C}\}$.

The cost function of each agent depends on the system state and control input. The costs of different agents are coupled through a mean-field term that depends on the mean of the population state. We write it as follows:

$$J_i(x_i, u_i, \bar{x}) = \mathbb{E}\left(V_i(x_i, u_i) + F(\bar{x}) \cdot x_i + G(\bar{x})\right), \tag{2}$$

where $\bar{x} \in \mathcal{X}$ is the average of the population state, i.e., $\bar{x} \triangleq \dfrac{1}{N}\sum_{i=1}^N x_i$, $F : \mathcal{X} \to \mathcal{X}$ is the mean-field coupling term, and $G : \mathcal{X} \to \mathbb{R}$ is the cost associated with the mean-field term. We impose the following regularity conditions on J_i.

Assumption 3: (i) $F(\cdot)$ is globally Lipschitz continuous on $\mathcal{X}$ with constant L , (ii) $G(\bar{x})$ is Fréchet differentiable on $\mathcal{X}$, and the gradient of $G(\bar{x})$ at 0 is bounded, i.e., $||\nabla G(0)|| < \infty$. (iii) the gradient of $G(\cdot)$ is globally Lipschitz continuous on $\mathcal{X}$ with constant β, i.e., $||\nabla G(x) - \nabla G(y)|| \leq \beta||x - y||$, $\forall x, y \in \mathcal{X}$.

Given these condition, the mean-field game considered in this paper is as follows:

$$\min_{u_i} \mathbb{E}\left(V_i(x_i, u_i) + F(\bar{x}) \cdot x_i + G(\bar{x})\right) \tag{3a}$$

$$\text{s.t. } x_i = f_i(u_i, \pi_i), \quad u_i \in \bar{\mathcal{U}}_i, \tag{3b}$$

where $\bar{x} \triangleq \dfrac{1}{N}\sum_{i=1}^N x_i$.

Remark 1: The mean-field game (3) captures a large class of problems studied in the literature. As it is formulated in general vector spaces, it includes both discrete-time [7], [18] and continuous-time system [3] as special cases, and addresses both deterministic and stochastic cases. The inner product term $F(\bar{x}) \cdot x_i$ in (3) is common in the literature. It either arises from the price multiplied by quantity term [8], [17] [19], or as part of the quadratic penalty of the deviation of the system state from the population mean [3], [12]. The structure of objective function (3) captures a large body of problems that frequently arise in various applications [3], [7], [12], [18], [20], [21].

Example 1: As an example, consider a deterministic mean-field game in discrete-time [7]. The objective function of the game is:

$$\min_{x_i} ||x_i||_Q^2 + ||x - \bar{x}||_\Delta^2 + 2(C\bar{x} + c)^T x_i \tag{4}$$

where $x_i \in \mathbb{R}^s$ is the system state, $\bar{x} = \dfrac{1}{N}\sum_{i=1}^N x_i$ is the average state, $||x_i||_Q^2$ stands for $x_i^T Q x_i$, $C \in \mathbb{R}^{s\times s}$, and Q and Δ are symmetric positive definite matrices of appropriate dimensions. Next, we show that the above game problem can be formulated as (3). Note that (4) is the degenerate case of (3), where the disturbance term π_i is 0, and $x_i = u_i$, i.e., f_i is the identity function. Expand the norm and combine similar terms in (4), then (4) is transformed to the following form:

$$\min_{x_i} x_i^T(Q + \Delta)x_i + 2c^T x_i + 2\bar{x}^T(C - \Delta)x_i + \bar{x}^T\Delta\bar{x} \tag{5}$$

Comparing (5) to (3), we have $V_i(x_i, u_i) = x_i^T(Q + \Delta)x_i + 2c^T x_i$, $F(\bar{x}) = 2(C - \Delta)\bar{x}$, and $G(\bar{x}) = \bar{x}^T\Delta\bar{x}$. In this case, it is easy to verify that the game problem (5) satisfies Assumption 1-3. Therefore, (5) is a special case of our proposed mean-field game (3).

Aside from this example, many continuous-time stochastic games are also special cases of (3), including the seminal work by Huang *et al.* [3]. More examples can be found in [17] and [22].

The objective of this paper is to study the mean-field game (3) from the optimization perspective: we aim to connect the mean-field game (3) to an auxiliary optimization problem. Once the connection is established, we can study the properties of the mean-field equilibrium by looking at the solution to the auxiliary optimization problem. For the latter, many powerful tools are available in the literature. This leads to new results on the existence, uniqueness and computation of the mean-field equilibrium.

III. Connection to the Auxiliary Optimization

This section studies the mean-field equilibrium of (3) in two steps. First, we characterize the mean-field equilibrium as the solution of a set of equations, i.e., mean-field equations. Second, we show that the mean-field equations have the same solution to an auxiliary optimization problem.

A. The Mean-Field Equations

In the context of mean-field games, we usually relax the Nash equilibrium solution concept by assuming that each agent is indifferent to an arbitrarily small change ϵ. This solution concept is referred to as the ϵ-Nash equilibrium, formally defined as follows:

Definition 1: $(u_1^*, \ldots, u_N^*)$ is an ϵ-Nash equilibrium of the game (3) if the following inequality holds

$$J_i(u_i^*, u_{-i}^*) \leq J_i(u_i, u_{-i}^*) + \epsilon \tag{6}$$

for all $i = 1, \ldots, N$, and all $u_i \in \bar{\mathcal{U}}_i$, where $u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$ and $J_i(u_i, u_{-i})$ is the compact notation for (2) after plugging (1) in (2).

At an ϵ-Nash equilibrium, each agent can lower his cost by at most ϵ via deviating from the equilibrium strategy, given that all other players follow the equilibrium strategy. When $\epsilon = 0$, the ϵ-Nash equilibrium reduces to a Nash equilibrium. Therefore, we can regarded it as a relaxed Nash equilibrium.

To derive the ϵ-Nash equilibrium of (3), we note that the utility function (2) is only coupled through the mean field term $F(m)$ and $G(m)$, and each agent has an neglectible impact on the coupling term as the population size is large. Therefore, in the large population case, we can approximate the agent's behavior with the optimal response to a deterministic value $y \in \mathcal{X}$ that replaces the mean field term $F(m)$ in the utility function (2). To this end, we define the following optimal response problem:

$$\mu_i(y) = \arg\min_{u_i} \mathbb{E}\left(V_i(x_i, u_i) + y \cdot x_i\right) \tag{7}$$

subject to:

$$\begin{cases} x_i = f_i(u_i, w_i) \\ x_i \in \mathcal{X}_i, \quad u_i \in \bar{\mathcal{U}}_i, \end{cases} \tag{8}$$

where $\mu_i(y)$ denotes the optimal solution to the optimization problem (7) parameterized by y, and $G(m)$ is regarded as a constant in (7) that can be ignored. Ideally, the deterministic mean field term approximation y guides the individual agents to choose a collection of optimal responses $\mu_i(y)$ which, in return, collectively generate the mean field term $\frac{1}{N}\sum_{i=1}^N f_i(\mu_i(y), w_i)$, and this should be close to y. In other words, we suggests to use following equation systems to characterize the equilibrium of the mean field game:

$$\begin{cases} \mu_i(y) = \arg\min_{u_i \in \bar{\mathcal{U}}_i} \mathbb{E}\left(V_i(x_i, u_i) + y \cdot x_i\right) & (9) \\ x_i^* = f_i(\mu_i(y), w_i) & (10) \\ y = F\left(\frac{1}{N}\mathbb{E}\sum_{i=1}^N x_i^*\right), & (11) \end{cases}$$

Formally, we can show that the solution to (9)-(11) is an ϵ-Nash equilibrium of the mean field game (3).

Theorem 1: The solution to the equation system (9)-(11), is an ϵ_N-Nash equilibrium of the mean field game (3), and $0 < \epsilon = O(\frac{1}{\sqrt{N}})$.

The proof of Theorem 1 can be found in [17]. It indicates that each agent is motivated to follow the equilibrium strategy u_i^* as deviating from this strategy can only improve the individual utility by a negligible amount ϵ. Furthermore, this ϵ can be arbitrarily small, if the population size is sufficiently large.

B. Connection to the Auxiliary Optimization

In the literature, a natural candidate for the auxiliary optimization is the sum of individual costs. This gives rise to the following social welfare optimization problem:

$$\min_{(u_1, \ldots, u_N)} \sum_{i=1}^N \mathbb{E}\left(V_i(x_i, u_i) + F(\bar{x}) \cdot x_i + G(\bar{x})\right) \tag{12}$$

$$\text{s.t. } x_i = f_i(u_i, \pi_i), \quad u_i \in \bar{\mathcal{U}}_i, \quad \forall i = 1, \ldots, N. \tag{13}$$

As the cost function (12) represents the social welfare of the entire system, it is desirable to attain the mean-field equilibrium at the solution to (12). However, it is proved in the literature that the mean-field equilibrium is not efficient. [3], [15], [16].

Different from the literature, we construct an auxiliary optimization problem by dropping the mean-field terms and introducing a virtual agent with the cost function $\phi : \mathcal{X} \to \mathbb{R}$. This auxiliary optimization problem is written as follows:

$$\min_{u_1, \ldots, u_N, z} \mathbb{E}\left(\sum_{i=1}^N V_i(x_i, u_i) + \phi(z)\right) \tag{14}$$

$$\text{s.t.} \begin{cases} z = \frac{1}{N}\sum_{i=1}^N \mathbb{E}x_i, \\ x_i = f_i(u_i, \pi_i), \quad u_i \in \bar{\mathcal{U}}_i, \quad \forall i = 1, \ldots, N \end{cases} \tag{15}$$

where z is the decision of the virtual agent. In the rest of this subsection, we show that when the virtual cost satisfies some conditions, the mean-field equations and the auxiliary optimization (14) have the same solution.

For our purpose, we first introduce the concept of strong duality for the auxiliary optimization problem (14). For notation convenience, we compactly denote (14) as follows:

$$P^* = \min_{u,z} J_s(u, z) \tag{16}$$

$$\text{s.t.} \begin{cases} g(u, z) = 0 \\ z \in \mathcal{X}, u_i \in \bar{\mathcal{U}}_i, \quad \forall i = 1, \ldots, N, \end{cases} \tag{17}$$

where P^* is the optimal value of the (16), $u = (u_1, \ldots, u_N)$ is the vector of control inputs, $J_s(u,z) = \mathbb{E}\left(\sum_{i=1}^N V_i(f_i(u_i,\pi_i),u_i) + \phi(z)\right)$, and $g(u,z) = \mathbb{E}\sum_{i=1}^N f_i(u_i,\pi_i) - Nz$. Then the Lagrangian of problem (16) can be defined as follows:

$$L(u,z,\lambda) = J_s(u,z) + \lambda \cdot g(u,z). \tag{18}$$

where $\lambda \in \mathcal{X}$ is the Lagrange multiplier for the constraint $g(u,z) = 0$. Define a mapping $D : \mathcal{X} \to \mathbb{R}$ as follows:

$$D(\lambda) = \inf_{u_i \in \bar{\mathcal{U}}_i, z \in \mathcal{X}} L(u,z,\lambda), \tag{19}$$

then the dual problem of the auxiliary optimization problem (14) is:

$$D^* = \max_{\lambda \in \mathcal{X}} D(\lambda) \tag{20}$$

where D^* is the optimal value of the dual problem. The auxiliary optimization problem has strong duality, if the dual problem (20) admits a solution, and its optimal value coincides with that of the primal problem (16).

Definition 2: The optimization problem (14) has strong duality if $P^* = D^*$ and there exists $\lambda^* \in \mathcal{X}$ such that $D^* = D(\lambda^*)$.

Under strong duality, we can establish connections between the mean-field equilibrium and the auxiliary optimization problem. This is summarized in the following two theorems:

Theorem 2: Let $\phi : \mathcal{X} \to \mathbb{R}$ be a Fréchet differentiable functional such that $\nabla\phi(z) = NF(z), \forall z \in \mathcal{X}$. Assume that (14) has strong duality, then any optimal solution to (14) is a mean-field equilibrium to the game (3).

Proof: Since the social welfare optimization (14) has strong duality, then there exists λ^* such that $P^* = D^* = D(\lambda^*)$. Note that due to weak duality, this indicates that λ^* is the optimal solution to the dual problem (20), i.e., $D^* = \inf_{u_1 \in \bar{\mathcal{U}}_1, \ldots, u_N \in \bar{\mathcal{U}}_N, z \in \mathcal{X}} L(u,z,\lambda^*)$. Let (u^*, z^*) be the optimal solution to (14), then (u^*, z^*) satisfies the constraint $z^* = \frac{1}{N}\sum_{i=1}^N \mathbb{E}f(u_i^*, \pi_i)$, and we have the following inequalities:

$$\begin{aligned} D^* &= \inf_{u_1 \in \bar{\mathcal{U}}_1, \ldots, u_N \in \bar{\mathcal{U}}_N, z \in \mathcal{X}} L(u,z,\lambda^*) \\ &\leq L(u^*, z^*, \lambda^*) = J_s(u^*, z^*) + \lambda^* \cdot g(u^*, z^*) \\ &= J_s(u^*, z^*) = P^*. \end{aligned} \tag{21}$$

Due to strong duality, $P^* = D^*$. Therefore, equality holds in (21), indicating that (u^*, z^*) satisfies the following:

$$(u^*, z^*) \in \operatorname*{arg\,min}_{u_1 \in \bar{\mathcal{U}}_1, \ldots, u_N \in \bar{\mathcal{U}}_N, z \in \mathcal{X}} L(u,z,\lambda^*). \tag{22}$$

Since L can be decomposed in terms of u^i and z, the relation (22) is equivalent to the following:

$$\begin{cases} u_i^* \in \operatorname*{arg\,min}_{u_i \in \bar{\mathcal{U}}_i} \mathbb{E}\left(V_i(f_i(u_i,\pi_i),u_i) + \lambda^* \cdot f_i(u_i,\pi_i)\right) & (23) \\ z^* \in \operatorname*{arg\,min}_{z \in \mathcal{X}} \phi(z) - N\lambda^* \cdot z & (24) \end{cases}$$

The first-order optimality condition of (24) yields $\nabla\phi(z^*) = N\lambda^*$. Since $\nabla\phi(z) = NF(z)$, we have $F(z^*) = \lambda^*$. Therefore, the above equation sets can be reduced to the following:

$$\begin{cases} u_i^* \in \operatorname*{arg\,min}_{u_i \in \bar{\mathcal{U}}_i} \mathbb{E}\left(V_i(f_i(u_i,\pi_i),u_i) + \lambda^* \cdot f_i(u_i,\pi_i)\right) & (25) \\ \lambda^* = F\left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}f_i(u_i^*,\pi_i)\right) & (26) \end{cases}$$

It can be verified that (25)-(26) is equivalent to the mean-field equations (9)-(11). Therefore, $(u_1^*, \ldots, u_N^*)$ is a mean-field equilibrium. This completes the proof. ■

Theorem 2 shows that any solution to the auxiliary optimization problem is a mean-field equilibrium. Note that this relation only holds from one direction: it does not necessarily mean that any mean-field equilibrium is also the optimal solution to (14).

However, the other direction also holds if the following monotonicity condition is imposed on the mean-field coupling term $F(\cdot)$:

Definition 3 (monotone mean-field coupling): The mean-field coupling term $F(x)$ is monotone with respect to $x \in \mathcal{X}$, if $(F(x) - F(x')) \cdot (x - x') \geq 0$ for any $x, x' \in \mathcal{X}$.

Under this monotone assumption, we have the following theorem:

Theorem 3: Let $\phi : \mathcal{X} \to \mathbb{R}$ be a Fréchet differentiable functional such that $\nabla\phi(z) = NF(z), \forall z \in \mathcal{X}$. Assume that the auxiliary optimization problem (14) has strong duality, and assume that $F(\cdot)$ is monotone, then $(u_1^*, \ldots, u_N^*)$ is the mean-field equilibrium to (3) if and only if it is the globally optimal solution to the auxiliary optimization problem (14).

Proof: Based on Theorem 2, the optimal solution to (14) is a mean-field equilibrium. Therefore, it suffices to show the other direction also holds. For notational convenience, let $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_N)$ be the solution to the mean-field equations (9)-(11). Define $\bar{z} = \frac{1}{N}\mathbb{E}\sum_{i=1}^N f_i(\bar{u}_i, \pi_i)$, and let $\bar{y} = F(\bar{z})$. Since $F(z) = \frac{1}{N}\nabla\phi(z)$, we have $\nabla\phi(\bar{z}) = N\bar{y}$. Since $F(z)$ is monotone, $\phi(z)$ is convex. Therefore, $\nabla\phi(\bar{z}) = N\bar{y}$ indicates that:

$$\bar{z} \in \operatorname*{arg\,min}_{z \in \mathcal{X}} \phi(z) - N\bar{y} \cdot z. \tag{27}$$

Due to (9), we also have:

$$\bar{u}_i \in \operatorname*{arg\,min}_{u_i \in \bar{\mathcal{U}}_i} \mathbb{E}\left(V_i(f_i(u_i,\pi_i),u_i) + \bar{y} \cdot f_i(u_i,\pi_i)\right). \tag{28}$$

The above two equations together indicate that $(\bar{u}, \bar{z})$ is the optimal solution to the following optimization problem:

$$\min_{u,z} \sum_{i=1}^N \mathbb{E}V_i(x_i,u_i) + \phi(z) + \bar{y} \cdot (\sum_{i=1}^N \mathbb{E}x_i - Nz) \tag{29}$$

$$\text{s.t.} \begin{cases} x_i = f_i(u_i,\pi_i) \\ u_i \in \bar{\mathcal{U}}_i, z \in \mathcal{X}. \end{cases} \tag{30}$$

In other words, $(\bar{u}, \bar{z})$ satisfies:

$$(\bar{u}, \bar{z}) \in \operatorname*{arg\,min}_{u_1 \in \bar{\mathcal{U}}_1, \ldots, u_N \in \bar{\mathcal{U}}_N, z \in \mathcal{X}} L(u,z,\bar{y}). \tag{31}$$

Note that due to weak duality, we have:

$$L(\bar{u},\bar{z},\bar{y}) \leq D^* \leq P^*. \quad (32)$$

On the other hand, we also have:

$$\begin{aligned} L(\bar{u},\bar{z},\bar{y}) &= J_s(\bar{u},\bar{z}) + \bar{y}\cdot g(\bar{u},\bar{z}) \\ &= J_s(\bar{u},\bar{z}) \geq P^*, \end{aligned} \quad (33)$$

where the last inequality is due to the fact that P^* is the minimum value of $J_s(u,z)$ among all (u,z) such that $z = \frac{1}{N}\mathbb{E}\sum_{i=1}^N f_i(u_i,\pi_i)$, and $(\bar{u},\bar{z})$ is one of them. Combing (32) and (33), we have $L(\bar{u},\bar{z},\bar{y}) = P^*$, thus $(\bar{u},\bar{z})$ is the globally optimal solution to (14). This completes the proof. ■

This theorem establishes equivalence between the mean-field equilibrium and the optimal solutions to (14). When the mean-field term is monotone, the solution set of the mean-field equations is the same as that of the auxiliary optimization problem as long as (14) has strong duality. Regarding this result, an interesting special case is when (14) is convex. To ensure convexity, we introduce the following conditions:

Assumption 4: (i) $f_i(u_i,\pi_i)$ is affine with respect to u_i, $\forall \pi_i \in \Pi$, (ii) $V_i(x_i,u_i)$ is convex with respect to (x_i,u_i), (iii) $\mathcal{X}$ and $\mathcal{U}_i$ are convex, (iv) $F(\cdot)$ is monotone.

Under Assumption 4, the auxiliary optimization problem has strong duality under mild constraint qualifications. Therefore, we have the following corollary:

Corollary 1: Let $\phi : \mathcal{X} \to \mathbb{R}$ be a Fréchet differentiable functional such that $\nabla\phi(z) = NF(z), \forall z \in \mathcal{X}$. Assume that the interior of the set $\mathcal{U}_i$ is non-empty, and the conditions of Assumption 4 hold, then $(u_1^*,\ldots,u_N^*)$ is the mean-field equilibrium to (3) if and only if it is the globally optimal solution to the auxiliary optimization problem (14).

It can be verified that Corollary 1 is equivalent to the result in [17]. This paper is different from [17] in that we relax the convexity assumption for the auxiliary optimization problem. This generalizes to the case where the auxiliary optimization is non-convex but has strong duality. The following is an example of this case.

Example 2: Consider a game with N agents. Each agent i wants to minimize the following objective function:

$$\min_{x_i\in\mathbb{R}} (x_i-\alpha_i)^2 + \kappa x_i \sin\left(\bar{x} - \frac{1}{N}\sum_{i=1}^N \alpha_i\right), \quad (34)$$

where α_i is a scalar and $\kappa > 0$ is a positive constant. In this example, the mean-field coupling term is $F(\bar{x}) = \kappa\sin\left(\bar{x} - \frac{1}{N}\sum_{i=1}^N \alpha_i\right)$. It is clear that the monotonicity condition does not hold.

Consider a virtual supply cost that satisfies $\nabla\phi(z) = NF(z)$. This gives $\phi(z) = -N\kappa\cos\left(z - \sum_{i=1}^N \alpha_i\right)$, then the auxiliary optimization is as follows:

$$\min_{x_1,\ldots,x_N,z} \sum_{i=1}^N (x_i-\alpha_i)^2 - N\kappa\cos\left(z - \frac{1}{N}\sum_{i=1}^N \alpha_i\right) \quad (35)$$

$$\text{s.t.} \begin{cases} z = \frac{1}{N}\sum_{i=1}^N x_i; \\ x_i = \mathbb{R}, z \in \mathbb{R}, \quad \forall i = 1,\ldots,N. \end{cases} \quad (36)$$

The above problem is non-convex with respect to $(x_1,\ldots,x_N,z)$, but we can still show that the primal problem (35) has strong duality. In particular, we summarize the result in the following theorem:

Theorem 4: If $0 < \kappa < 2$, then (i) the auxiliary optimization (35) has strong duality, (ii) the mean-field equations for (34) has a unique solution, (iii) the decisions $(x_1^*,\ldots,x_N^*)$ is the mean-field equilibrium for (34) if and only if it is the globally optimal solution to the auxiliary optimization (35).

Proof: To prove (i), it suffices to show that there exists λ^* such that $P^* = D^* = D(\lambda^*)$. To show this, we first note that since $\kappa > 0$, the cost function of primal problem (35) is lower bounded by $-N\kappa$:

$$\sum_{i=1}^N (x_i-\alpha_i)^2 - N\kappa\cos\left(z - \frac{1}{N}\sum_{i=1}^N \alpha_i\right) \geq -N\kappa$$

It can be verified that when $x_i = a_i$ and $z = \frac{1}{N}\sum_{i=1}^N \alpha_i$, the cost function of (35) equals $-N\kappa$ and the constraints are satisfied. Therefore, $-N\kappa$ is the optimal value for the primal problem. According to Definition 2, it suffices to find λ^* such that the minimum value of $L(u,z,\lambda^*)$ is also $-N\kappa$. Let $\lambda^* = 0$, then the Lagrangian dual $L(u,z,0)$ corresponds to the following problem:

$$\min_{x_1,\ldots,x_N,z} \sum_{i=1}^N (x_i-\alpha_i)^2 - N\kappa\cos\left(z - \frac{1}{N}\sum_{i=1}^N \alpha_i\right) \quad (37)$$

$$\text{s.t.: } x_i = \mathbb{R}, z \in \mathbb{R}, \quad \forall i = 1,\ldots,N..$$

The optimal value of (37) is clearly $-N\kappa$. Therefore, strong duality holds.

To prove (ii), the idea is to construct a contraction mapping, whose fixed point is the solution to the mean-field equations. In particular, we first regard m as given and solve the problem (34) to derive the optimal solution as $x_i^* = \alpha_i - \frac{1}{2}\kappa\sin\left(\bar{x} - \frac{1}{N}\sum_{i=1}^N \alpha_i\right)$. Then we define a function $T : \mathbb{R} \to \mathbb{R}$ that maps $\bar{x}$ to the average of x_i^*:

$$T(\bar{x}) = \frac{1}{N}\sum_{i=1}^N \alpha_i - \frac{1}{2}\kappa\sin\left(\bar{x} - \frac{1}{N}\sum_{i=1}^N \alpha_i\right). \quad (38)$$

It can be verified that the mean-field equilibrium is the fixed point of this mapping. Since $|\sin(x)-\sin(y)| \leq |x-y|$, we have $T(m_1) - T(m_2) \leq \frac{1}{2}\kappa|m_1 - m_2|$ for any $m_1 \in \mathbb{R}$ and $m_2 \in \mathbb{R}$. Therefore, as $\kappa < 2$, $T(\bar{x})$ is a contraction mapping, and it has a unique fixed point.

To prove (iii), we can use result (i), (ii) and Theorem 2. This completes the proof. ■

Remark 2: The existence and uniqueness of (34) are derived based on the connection between the mean-field game and the auxiliary optimization problem. Note that this is not possible based on results in the existing literature, where the existence of mean-field equilibria typically requires the agent cost function to be convex with respect to control [1], [6], [23], [24], and the uniqueness typically requires the mean-field term $F(\cdot)$ to be monotone [6], [24], [25]. Neither of these conditions holds for (34).

IV. Relation to Potential Game

Theorem 2 and Theorem 3 shows that the mean-field equilibrium is the solution to the auxiliary optimization problem. From this perspective, the auxiliary optimization is similar to potential functions in a potential game, where the Nash equilibrium is the optimal solution to the potential function minimization [26]. However, we emphasize that in general, the mean-field game (3) considered in this paper is not a potential game, and the auxiliary optimization is not a potential function. In this section, we discuss the differences between the auxiliary optimization and the potential function minimization in terms of solution concept and computation complexity.

A. A Counter Example

In general, the mean-field game (3) is not a potential game. As an example, consider a game with the following objective function:

$$J_i(x_1, \cdots, x_N) = (x_i - 1)^2 + x_i \log \bar{x}, \tag{39}$$

According to Theorem 4.5 in [27], the necessary and sufficient condition for (39) to be a potential game is that

$$\frac{\partial^2 J_i}{\partial x_i \partial x_j} = \frac{\partial^2 J_j}{\partial x_i \partial x_j}.$$

This clearly does not hold for (39). Therefore, the mean-field game (3) does not admit a potential function. On the one hand, it is easy to verify that (39) satisfies Assumption 4. Therefore, based on Theorem 3, we can still use the auxiliary optimization (14) to characterize the mean-field equilibrium.

B. Compare Auxiliary Optimization with Potential Function Minimization

Under some additional assumptions (i.e., $F(\bar{x})$ is linear function), the mean-field game (3) may reduce to a potential game. In this case, we can either use the auxiliary optimization or the potential function to study the mean-field equilibrium. These two methods provide different solutions. In this subsection, we compare them in terms of solution concepts and computation complexity.

For illustration purpose, consider a game problem with the following objective function:

$$J_i(x_1, \cdots, x_N) = (x_i - 1)^2 + x_i \bar{x}. \tag{40}$$

On the one hand, it can be verified that (40) is a potential game, and the potential function minimization problem is:

$$\min_{x_1,\ldots,x_N} \sum_{i=1}^{N} (x_i - 1)^2 + \frac{1}{N} \sum_{i=1}^{N} + \frac{1}{2N} \sum_{i \neq j} x_i x_j. \tag{41}$$

On the other hand, the objective function (40) satisfies Assumption 4. According to Corollary 1, the mean-field equilibrium is the solution to the following auxiliary optimization problem:

$$\min_{x_1,\ldots,x_N} \sum_{i=1}^{N} (x_i - 1)^2 + \frac{N}{2} \bar{x}^2. \tag{42}$$

Both (41) and (42) are convex optimization, and their solutions are $x_i = \dfrac{2N}{3N+1}$ and $x_i = \dfrac{2}{3}$, respectively.

These solutions are both equilibria of the mean-field game (40). Their differences are as follows: first, potential function provides an exact Nash equilibrium to (40), while the auxiliary optimization produces an ϵ-Nash equilibrium. This is a relaxed solution concept compared to Nash equilibrium. Second, the potential function minimization is typically solved as a centralized optimization, but the auxiliary optimization can be implemented in a decentralized fashion [17]. This enables the parallel execution of the algorithm, thus the time complexity for solving (14) does not depend on N. To summarize, the auxiliary optimization method achieves better scalability than potential function minimization at the price of a relaxed solution concept.

V. Conclusion

This paper studies a class of mean-field games from the optimization perspectives. We showed that the ϵ-Nash equilibrium of the mean-field game is the optimal solution to an auxiliary optimization problem. This connection enables us to derive the mean-field equilibrium by solving the corresponding optimization problems. This optimization problem is different from the potential function minimization, and we compared them in terms of solution concepts and computation complexity. Future work includes extending the proposed approach to the case of infinitely many agents and more general formulations where the mean field term depends on the probability distribution of the population state.

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