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Brayton–Moser Formulation of Distributed Port-Hamiltonian Systems Modelling and Boundary Control

Alessandro Macchelli
Department of of Electrical, Electronic and Information Engineering
University of Bologna, Italy
Email: alessandro.macchelli@unibo.it

***Abstract*— In this paper, for a class of linear, distributed port-Hamiltonian systems defined on a one-dimensional spatial domain, an equivalent Brayton–Moser formulation is obtained. The dynamic is expressed as a gradient equation with respect to a new storage function, the "mixed-potential," with the dimensions of power, and the system is passive with respect to a supply rate related to the reactive power. The Brayton–Moser representation is the starting point for the synthesis of boundary control laws whose effect on the system is to shape the mixed-potential function. The general theory is illustrated with the help of an example, namely the boundary stabilisation of longitudinal vibrations of a beam with internal dissipation.**

***Index Terms*—**

I. Introduction

Port-Hamiltonian systems [1] can be considered as the mathematical formalization of bond-graphs to describe lumped parameter physical systems, [2], [3]. The generalization to the infinite dimensional scenario leads to the definition of distributed port-Hamiltonian systems [4], [5] that represent a particular case of the more general framework presented in [6] and then extended in [7] to open physical systems. Distributed port-Hamiltonian systems have proved to represent a powerful framework for modeling, simulation and control of physical systems described by PDEs.

Most of the current research on the stabilisation of distributed port-Hamiltonian systems is about the development of boundary controllers. The simplest synthesis methodology consists in adding some dissipation, and / or in shaping the energy function to shift the equilibrium, with stability being ensured by the passivity properties of the controlled system, and proved by using the total energy (Hamiltonian) as Lyapunov function, [7]–[15]. Here, the idea is to procede in a different manner by determining a control action able to shape a *power-related* storage function. In other words, in this paper the generalisation of the power-based control of finite dimensional port-Hamiltonian systems [16]–[18] to the distributed parameter case is presented.

In this respect, the first step consists in re-writing the port-Hamiltonian dynamic in the so-called Brayton–Moser form, i.e. in terms of a gradient equation of a mixed-potential function with the unity of power, [19]. Originally developed for the description of RLC networks, the Brayton–Moser equations have been extended to nonlinear systems in [16]–[18], and to the description of autonomous distributed parameter systems in [20]. Here, the relation between such representation and the port-Hamiltonian formalism is presented, and exploited for control synthesis purposes. This approach differs from the port-Hamiltonian theory because the candidate Lyapunov function is not the Hamiltonian (i.e., the total energy), but the mixed-potential (i.e., a quantity related to the power distribution). The main advantage is that in the proposed stabilisation procedure the internal dissipation does not pose any constraint on the applicability of the method itself. In fact, it is well known that control schemes that rely on energy-balancing are limited by the presence of pervasive dissipation: this issue is usually called *dissipation obstacle* in literature, [21]. Moreover, control techniques based on energy-shaping and able to overcome this constraint, require to have at disposal the full-state of the plant [8]–[10], [14].

The paper is organised as follows. In Section II, the class of distributed port-Hamiltonian systems for which the presented result are applicable is briefly discussed, together with their main properties. Their equivalent formulation in terms of a Brayton–Moser equation is presented in Section III, and then exploited for the development of buondary stabilising control laws in Section IV. The proposed methodologies are illustrated with the help of an example, the buondary control of the transversal vibrations in an elastic beam, in Section V. Finally, conclusions and indications about possible future developments and open problems are reported in Section VI.

II. Distributed port-Hamiltonian systems

In this paper, we refer to a class of distributed port-Hamiltonian systems described by the PDE

$$\frac{\partial x}{\partial t}(t,z) = P_1 \frac{\partial}{\partial z}\frac{\delta \mathcal{H}}{\delta x}(x(t,z)) + \big[P_0(z) - G_0(z)\big]\frac{\delta \mathcal{H}}{\delta x}(x(t,z)) \tag{1}$$

with $x \in X := L^2(a, b; \mathbb{R}^n)$, and $z \in [a, b]$. Moreover, $P_1 = P_1^{\mathrm{T}}$ and invertible, $P_0(z) = -P_0^{\mathrm{T}}(z)$, $G_0(z) = G_0^{\mathrm{T}}(z) \geq 0$, and $\mathcal{H}(x)$ is the Hamiltonian function, bounded from below. It is also assumed that

$$\mathcal{H}(x(t,\cdot)) = \int_a^b H(x(t,z),z)\,\mathrm{d}z \tag{2}$$

being $H(x, z)$ the energy density. Such class is an extension of the linear case studied in [14], [15], [22], [23], which corresponds to a quadratic energy density. Finally, in (1), δ denotes the functional gradient, i.e. the variational derivative, [4]–[6]. In the next lemma, it is shown how to compute it

for a simpler class of functionals employed in this paper and of which the Hamiltonian (2) is a particular case.

Lemma 2.1: Let us consider the functional

$$\mathcal{F}(f) = \int_a^b F(f, f_z)\,\mathrm{d}z$$

where F is a C^1 function, f is a C^1 function such that $f(a) = f(b) = 0$, and $f_z = \frac{\partial f}{\partial z}$. Then,

$$\begin{aligned}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t}(f) &= \int_a^b \frac{\partial^{\mathrm{T}} f}{\partial t}\left[\frac{\partial F}{\partial f}(f, f_z) - \frac{\partial}{\partial z}\frac{\partial F}{\partial f_z}(f, f_z)\right]\mathrm{d}z \\ &=: \int_a^b \frac{\partial^{\mathrm{T}} f}{\partial t}\frac{\delta \mathcal{F}}{\delta f}(f)\,\mathrm{d}z\end{aligned} \tag{3}$$

that implicitly defines the functional gradient (or variational derivative), of the functional $\mathcal{F}$.

Remark 2.1: The state space of (1) is called space of energy variables, while $\frac{\delta \mathcal{H}}{\delta x}$ denotes the co-energy variables. From the previous lemma, since the energy density $H(x, z)$ in (2) does not depend on the derivatives of the state variable x with respect to the spatial coordinate z, we have that

$$\frac{\delta \mathcal{H}}{\delta x}(x(t, z)) = \frac{\partial H}{\partial x}(x(t, z)) =: e(t, z) \tag{4}$$

i.e. the co-energy variables e are the partial derivative of the energy density H with respect to the energy variables x.

To define the boundary input and output for the PDE (1), at first the port variables f_∂, $e_\partial \in \mathbb{R}^n$ given by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=:R}\begin{pmatrix} \frac{\partial H}{\partial x}(b) \\ \frac{\partial H}{\partial x}(a) \end{pmatrix} \tag{5}$$

are introduced, [22]. They are a linear combination of the restriction of the co-energy variables to the boundary of the spatial domain, and integration by parts shows that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) \leq e_\partial^{\mathrm{T}}(t) f_\partial(t)$.

The problem of defining the boundary input for (1) that leads to a boundary control system on X in the sense of the semigroup theory, see e.g. [24], has been addressed in [22] in the linear case, i.e. when $H(x, z)$ is quadratic. More details on this point in Remark 2.2. In this paper, the same parametrisation is adopted for both the input and the output. In this respect, denote by W be a $n \times 2n$ full rank, real matrix, and define the input $u(t)$ as

$$u(t) = W\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{6}$$

Remark 2.2: Assume that H is quadratic, i.e. that here exists a bounded and continuous matrix-valued function $\mathcal{L}(\cdot)$ such that $\mathcal{L}(z) = \mathcal{L}^{\mathrm{T}}(z)$ and $\mathcal{L}(z) \geq \kappa I$, with $\kappa > 0$, for all $z \in [a,\, b]$, and $H(x, z) = \frac{1}{2}x^{\mathrm{T}}\mathcal{L}(z)x$. Clearly, $\frac{\delta \mathcal{H}}{\delta x} = \frac{\partial H}{\partial x} = \mathcal{L}x \in H^1(a, b; \mathbb{R}^n)$. As discussed in [22], since W has full rank and if $W\Sigma W^{\mathrm{T}} \geq 0$, with

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

then (1) with input (6) is a boundary control system on X, and the operator $\mathcal{J}x := P_1\frac{\partial}{\partial z}(\mathcal{L}x) + (P_0 - G_0)\mathcal{L}x$ with domain $D(\mathcal{J}) = \left\{\mathcal{L}x \in H^1(a, b; \mathbb{R}^n) \mid u = 0\right\}$ generates a contraction semigroup on X. Here, $H^1(a, b; \mathbb{R}^n)$ denotes the Sobolev space of order one.

Now, let $\tilde{W}$ be a $n \times 2n$ full rank, real matrix, such that $\begin{pmatrix} W^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible, and let P_W be given by

$$P_W = \begin{pmatrix} W\Sigma W^{\mathrm{T}} & W\Sigma\tilde{W}^{\mathrm{T}} \\ \tilde{W}\Sigma W^{\mathrm{T}} & \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} \end{pmatrix}^{-1} \tag{7}$$

If the output is defined as

$$y(t) = \tilde{W}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{8}$$

then, the following energy balance equation is satisfied:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}(x(t)) = &-\int_a^b \frac{\partial^{\mathrm{T}} H}{\partial x}(x)G_0(z)\frac{\partial H}{\partial x}(x)\,\mathrm{d}z + \\ &+ \frac{1}{2}\begin{pmatrix} u(t) \\ y(t) \end{pmatrix}^{\mathrm{T}} P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}\end{aligned} \tag{9}$$

This means that (1) is dissipative with storage function $\mathcal{H}(x)$ and quadratic supply rate defined by P_W.

III. Brayton–Moser formulation

Brayton–Moser equations originally appeared in the description of RLC circuits [19]. Differently from the Hamiltonian formulation where the energy variables are used (charge and magnetic flux), the dynamic is expressed in terms of the co-energy variables (voltages and currents). The result is a gradient equation with respect to a mixed-potential function P that in finite dimensions usually takes the form

$$Q(e(t))\dot{e}(t) = \frac{\partial P}{\partial e}(e(t)) + B(e(t))u(t) \tag{10}$$

where e denotes the co-energy variables, and u is the input. In [25], the relation between port-Hamiltonian and Brayton–Moser formulations has been established in the lumped parameter case, and then exploited e.g. in [18] for control purposes. The scope of this section is to obtain the Brayton–Moser formulation of (1) by re-writing it in the co-energy variables $e = \frac{\partial H}{\partial x}(x)$ defined by (4).

Now, let us assume that relation (4) is invertible, which implies that there exists a map from the co-energy to the energy variables in the form

$$x(t, z) = \frac{\partial H^\star}{\partial e}(e(t, z), z) \tag{11}$$

with $H^\star$ the Legendre transformation of H, given as

$$H^\star(e(t, z), z) = e^{\mathrm{T}}(t, z)x(t, z) - H(x(t, z), z)$$

Then, the port-Hamiltonian system (1) can be expressed in the co-energy variables, since e and x are related:

$$\frac{\partial x}{\partial t}(t, z) = \frac{\partial^2 H^\star}{\partial e^2}(e(t, z), z)\frac{\partial e}{\partial t}(t, z) \tag{12}$$

Remark 3.1: When (1) is linear, i.e. when H is quadratic in the energy variables, the mapping (11) can be easily computed. In fact, under the same conditions of Remark 2.2, if $H(x, z) = \frac{1}{2}x^{\mathrm{T}}\mathcal{L}(z)x$, then $e(t, z) = \mathcal{L}(z)x(t, z)$, which

implies that $H^\star(e,z) = \frac{1}{2}e^{\mathrm{T}}\mathcal{L}^{-1}(z)e$, that is well defined since $\mathcal{L}(z)$ is non-singular.

Differently from the port-Hamiltonian system (1) where the Hamiltonian $\mathcal{H}$ defined in (2) depends on the state variable x and *not* on its derivative with respect to the spatial coordinate z, the extension of (10) to the distributed parameter case requires the definition of a (class of) mixed-potential functionals $\mathcal{P}$ that depends on the co-energy variable e *and* its spatial derivative.

Proposition 3.1 ([26]): Denote by $M_1 = -M_1^{\mathrm{T}}$ an invertible skew-symmetric matrix, and suppose that $P_0 = G_0 = 0$ in (1). Then, (1) admits the Brayton–Moser form

$$Q(e(t,z),z)\frac{\partial e}{\partial t}(t,z) = \frac{\delta \mathcal{P}}{\delta e}(e(t,z)) \tag{13}$$

where

$$Q(e,z) = M_1 P_1^{-1}\frac{\partial^2 H^\star}{\partial e^2}(e,z) \tag{14}$$

$$\mathcal{P}(e) = \frac{1}{2}\int_a^b e^{\mathrm{T}}(z)M_1\frac{\partial e}{\partial z}(z)\,\mathrm{d}z \tag{15}$$

being $H^\star$ the co-energy density, i.e. the Legendre trasformation of H. Moreover, for the mixed-potential $\mathcal{P}$ defined in (14), the following balance relation holds true:

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}(e(t)) = \int_a^b \frac{\partial^{\mathrm{T}} e}{\partial t}Q(e,z)\frac{\partial e}{\partial t}\,\mathrm{d}z + \\ + \frac{1}{2}\begin{pmatrix}u(t)\\ y(t)\end{pmatrix}^{\mathrm{T}} M_W \begin{pmatrix}\dot u(t)\\ \dot y(t)\end{pmatrix} \tag{16}$$

where u and y are the boundary input and output defined in (6) and (8), respectively, and

$$M_W = \begin{pmatrix}W\\ \tilde W\end{pmatrix}^{-\mathrm{T}} R^{-\mathrm{T}}\Sigma_M R^{-1}\begin{pmatrix}W\\ \tilde W\end{pmatrix}^{-1} \tag{17}$$

with

$$\Sigma_M = \begin{pmatrix}M_1 & 0\\ 0 & -M_1\end{pmatrix}$$

Note that, differently from (7), $M_W = -M_W^{\mathrm{T}}$.

Proof: From Lemma 2.1, we have that

$$\begin{aligned}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} &= \frac{1}{2}\int_a^b\left(\frac{\partial^{\mathrm{T}} e}{\partial t}M_1\frac{\partial e}{\partial z} + e^{\mathrm{T}}M_1\frac{\partial}{\partial z}\frac{\partial e}{\partial t}\right)\mathrm{d}z \\ &= \int_a^b \frac{\partial^{\mathrm{T}} e}{\partial t}M_1\frac{\partial e}{\partial z}\,\mathrm{d}z + \frac{1}{2}\left[e^{\mathrm{T}}(z)M_1\dot e(z)\right]_a^b\end{aligned} \tag{18}$$

which implies that $\frac{\delta\mathcal{P}}{\delta e} = M_1\frac{\partial e}{\partial z}$. From (12), we can rewrite (1) as follows:

$$\frac{\partial^2 H^\star}{\partial e^2}(e(t,z),z)\frac{\partial e}{\partial t}(t,z) = P_1 M_1^{-1}\frac{\delta\mathcal{P}}{\delta e}(e(t,z))$$

Since P_1 is non-singular, we obtain the expression (14) for $Q(e,z)$. The term in (18) that depends on the boundary conditions of (1), and their time derivative is

$$\left[e^{\mathrm{T}}(z)M_1\dot e(z)\right]_a^b = \begin{pmatrix}e(b)\\ e(a)\end{pmatrix}^{\mathrm{T}}\Sigma_M\begin{pmatrix}\dot e(b)\\ \dot e(a)\end{pmatrix}$$

where from (6) and (8), we have that

$$\begin{pmatrix}u\\ y\end{pmatrix} = \begin{pmatrix}W\\ \tilde W\end{pmatrix} R\begin{pmatrix}e(b)\\ e(a)\end{pmatrix} \tag{19}$$

which clearly implies that

$$\left[e^{\mathrm{T}}(z)M_1\dot e(z)\right]_a^b = \\ = \begin{pmatrix}u\\ y\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}W\\ \tilde W\end{pmatrix}^{-\mathrm{T}} R^{-\mathrm{T}}\Sigma_M R^{-1}\begin{pmatrix}W\\ \tilde W\end{pmatrix}^{-1}\begin{pmatrix}\dot u\\ \dot y\end{pmatrix}$$

since $\begin{pmatrix}W^{\mathrm{T}} & \tilde W^{\mathrm{T}}\end{pmatrix}$ is invertible. Finally, the couple of relations (16) and (17) are a consequence of the fact that

$$\begin{aligned}\frac{\partial^{\mathrm{T}} e}{\partial t}(t,z)M_1\frac{\partial e}{\partial z}(t,z) &= \frac{\partial^{\mathrm{T}} e}{\partial t}(t,z)\frac{\delta\mathcal{P}}{\delta e}(e(t,z)) \\ &= \frac{\partial^{\mathrm{T}} e}{\partial t}(t,z)Q(e(t,z),z)\frac{\partial e}{\partial t}(t,z)\end{aligned}$$

■

Remark 3.2: For any invertible, skew symmetric matrix M_1, we have a mixed-potential $\mathcal{P}$ and a corresponding (but equivalent) Brayton–Moser formulation (13) of (1). Moreover, given $\mathcal{P}$ as defined in (15), then also $\mathcal{P}_\Psi(e) = \mathcal{P}(e) + \Psi(e(a),e(b))$ is an admissible mixed-potential functional associated with the same PDE (13). The addition of the function Ψ has an effect on the balance relation (16) only and, more precisely, on the part that involves the boundary terms. This fact is exploited in Section IV for control synthesis purposes.

With Remark 3.2 in mind, Proposition 3.1 is now extended to cope with the general case, i.e. when in (1) $P_0(z)$ and $G_0(z)$ are different from 0.

Proposition 3.2: Denote by $M_1 = -M_1^{\mathrm{T}}$ an invertible, skew-symmetric matrix. Then, the PDE (1) can be written in the Brayton–Moser form (13), in which $Q(e,z)$ is given in (14), and the mixed-potential $\mathcal{P}$ is:

$$\mathcal{P}(e) = \frac{1}{2}\int_a^b e^{\mathrm{T}}(z)\left[M_1\frac{\partial e}{\partial z}(z) + M_0(z)e(z)\right]\mathrm{d}z + \\ + \frac{1}{2}\begin{pmatrix}e(b)\\ e(a)\end{pmatrix}^{\mathrm{T}} M_\partial\begin{pmatrix}e(b)\\ e(a)\end{pmatrix} \tag{20}$$

with $M_\partial = M_\partial^{\mathrm{T}}$, if there exists $M_0(z) = M_0^{\mathrm{T}}(z)$ such that

$$P_1 M_1^{-1}M_0(z) = P_0(z) - G_0(z) \tag{21}$$

Moreover, for the mixed-potential (20), the balance relation

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t}(e(t)) = \int_a^b \frac{\partial^{\mathrm{T}} e}{\partial t}Q(e,z)\frac{\partial e}{\partial t}\,\mathrm{d}z + \\ + \frac{1}{2}\begin{pmatrix}u(t)\\ y(t)\end{pmatrix}^{\mathrm{T}}(M_W + M'_\partial)\begin{pmatrix}\dot u(t)\\ \dot y(t)\end{pmatrix} \tag{22}$$

holds with M_W defined in (17) and

$$M'_\partial = 2\begin{pmatrix}W\\ \tilde W\end{pmatrix}^{-\mathrm{T}} R^{-\mathrm{T}}M_\partial R^{-1}\begin{pmatrix}W\\ \tilde W\end{pmatrix}^{-1} \tag{23}$$

with $(M'_\partial)^{\mathrm{T}} = M'_\partial$, being u and y the boundary input and output of (1) introduced in (6) and (8), respectively.

Proof: By taking the time derivative of (20), from (18) we obtain that

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = \int_a^b \frac{\partial^{\mathrm{T}} e}{\partial t}\left[M_1\frac{\partial e}{\partial z} + M_0(z)e\right]\mathrm{d}z + \\ + \begin{pmatrix} e(b) \\ e(a)\end{pmatrix}^{\mathrm{T}}\left(\frac{1}{2}\Sigma_M + M_\partial\right)\begin{pmatrix}\dot{e}(b)\\ \dot{e}(a)\end{pmatrix} \quad (24)$$

which implies that

$$\frac{\delta\mathcal{P}}{\delta e}(e(t,z),z) = M_1\frac{\partial e}{\partial z}(t,z) + M_0(z)e(t,z) \quad (25)$$

From (12), the PDE (1) can be written as

$$\frac{\partial^2 H^\star}{\partial e^2}(e,z)\frac{\partial e}{\partial t} = \\ = P_1 M_1^{-1}\left\{M_1\frac{\partial e}{\partial z} + M_1 P_1^{-1}\left[P_0(z) - G_0(z)\right]e\right\}$$

which leads to the Brayton–Moser formulation (13) with $Q(e,z)$ given as in (14), provided that (21) holds. The power balance relation (22) for the mixed-potential (20) follows in the same way as in the proof of Proposition 3.1, with M'_∂ given as in (23) and obtained in the same way as M_W, see (17). ∎

An important class of port-Hamiltonian systems that satisfies the conditions of the previous proposition is investigated in the next corollary. The equation modelling the damped longitudinal vibrations in a beam discussed in Section V, as long as the Timoshenko beam equation [9], or the shallow water equation, [12], [27] belong to this class.

Corollary 3.1 ([26]): Let us consider the port-Hamiltonian system (1), and suppose that $n = 2N$, with $N \geq 1$. Denote by $x = (x_q, x_p)$ the state variable, with $x_q, x_p \in L^2(a,b;\mathbb{R}^N)$, and by $e = (e_q, e_p)$ the co-energy variable. If

$$P_1 = \begin{pmatrix} 0 & \bar{P}_1 \\ \bar{P}_1^{\mathrm{T}} & 0\end{pmatrix} \qquad P_0(z) = \begin{pmatrix} 0 & \bar{P}_0(z) \\ -P_0^{\mathrm{T}}(z) & 0\end{pmatrix}$$
$$G_0(z) = \begin{pmatrix}\bar{G}_q(z) & 0 \\ 0 & \bar{G}_p(z)\end{pmatrix}$$

where $\bar{P}_1$, $\bar{P}_0(z)$, $\bar{G}_q(z)$ and $\bar{G}_p(z)$ are $N\times N$ real matrices such that $\bar{P}_1$ is non-singular, $\bar{G}_q(z) = \bar{G}_q^{\mathrm{T}}(z) \geq 0$, and $\bar{G}_p(z) = \bar{G}_p^{\mathrm{T}}(z) \geq 0$, then this system admits a Brayton–Moser formulation (13) with

$$Q(e_q,e_p,z) = \begin{pmatrix} -I & 0 \\ 0 & I\end{pmatrix}\frac{\partial^2 H^\star}{\partial(e_q,e_p)^2}(e_q,e_p,z) \quad (26)$$

and

$$\mathcal{P}(e_q,e_p) = \frac{1}{2}\int_a^b \begin{pmatrix} e_q \\ e_p\end{pmatrix}^{\mathrm{T}}\Bigg[\underbrace{\begin{pmatrix} 0 & -\bar{P}_1 \\ \bar{P}_1^{\mathrm{T}} & 0\end{pmatrix}}_{=:M_1}\frac{\partial}{\partial z}\begin{pmatrix} e_q \\ e_p\end{pmatrix} + \\ + \underbrace{\begin{pmatrix}\bar{G}_q(z) & -\bar{P}_0(z) \\ -\bar{P}_0^{\mathrm{T}}(z) & -\bar{G}_p(z)\end{pmatrix}}_{=:M_0(z)}\begin{pmatrix} e_q \\ e_p\end{pmatrix}\Bigg]\mathrm{d}z$$

Remark 3.3: Corollary 3.1 is a generalisation of a similar result presented in [20], where the Brayton–Moser formulation of a transmission line with dissipation is presented. An analogous PDE describing the longitudinal vibrations in a beam is discussed in Section V to illustrate the control synthesis methodology. In this respect, note that, differently from the original port-Hamiltonian system with the balance relation (9), for the Brayton–Moser equation it is *not* possible to rely on balance relation (16) to prove stability of an equilibrium. Beside the fact that $\mathcal{P}$ is not necessarily bounded or characterised by a (local) minimum at the desired equilibrium, the matrix Q is not negative definite, so $\mathcal{P}$ is not decreasing along system trajectories when $u(t) = 0$. This problem and a possible solution that extends [16], [20] are discussed in the next section.

IV. Boundary control

The scope of this section is to show how to exploit Bayton–Moser formulation of (1) to determine a control law $u(t)$ that stabilises an equilibrium $x_\star \in L^2(a,b;\mathbb{R}^n)$ or, equivalently, $e_\star \in H^1(a,b;\mathbb{R}^n)$. Clearly, from (11) and (12), we have that

$$x_\star(z) = \frac{\partial H^\star}{\partial e}(e_\star(z),z) \quad \Leftrightarrow \quad \frac{\delta\mathcal{P}}{\delta e}(e_\star(z),z) = 0 \quad (27)$$

The main idea is to use the mixed-potential $\mathcal{P}$ as a Lyapunov function, and to rely on the balance relation (22) to compute a control action $u(t)$ so that $\dot{\mathcal{P}}(e(t))$ is not increasing along system trajectories. As in the finite dimensional case (see e.g. [16], [28]), asymptotic stability is then proved, for example, thanks to the LaSalle's Invariance Principle, [29, Theorem 3.64].

It has been pointed out in Remark 3.3 that the main limitation of the balance relation (22) which prevents its use in the control synthesis is that, in general, the symmetric part of the matrix Q is *not* negative semi-definite. For example, under the hypotheses of Corollary 3.1, from (26) we have that $Q + Q^{\mathrm{T}}$ is clearly sign indefinite. The goal is to determine, starting from a given mixed-potential $\mathcal{P}$, a *new* admissible functional $\mathcal{P}_a$ that satisfies a balance relation in the form (22) for some Q_a such that $Q_a + Q_a^{\mathrm{T}} \leq 0$. The problem of constructing a family of $\mathcal{P}_a$ and Q_a pairs among which selecting the ones with the desired property is takled in the next proposition, which is a simple extension of an analogous result presented in [20].

Proposition 4.1 ([26]): Let us consider the Brayton–Moser formulation (13) of the boundary control system (1) obtained in Proposition 3.2. Then, for any $\lambda \in \mathbb{R}$ and $n\times n$ symmetric matrix $\Lambda = \Lambda^{\mathrm{T}}$, if $M_1\Lambda Q(e,z)$ is symmetric, we have that

$$\mathcal{P}_a(e) = \lambda\mathcal{P}(e) + \frac{1}{2}\int_a^b \frac{\delta^{\mathrm{T}}\mathcal{P}}{\delta e}(e)\Lambda\frac{\delta\mathcal{P}}{\delta e}(e)\,\mathrm{d}z \quad (28)$$

and

$$Q_a(e,z) = \lambda Q(e,z) + M_0(z)\Lambda Q(e,z) + M_1\Lambda\frac{\partial Q(e,z)}{\partial z} \quad (29)$$

Proof: The result is proved by taking the time derivative of (28) under the constraint that the system evolution is expressed by (13), and $\frac{\delta \mathcal{P}}{\delta e}$ given by (25). As far as the first term in (28) is concerned, we have that $\dot{\mathcal{P}}$ is equal to (22) while, as far as the integral one, we can write that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\delta^{\mathrm{T}}\mathcal{P}}{\delta e}\Lambda\frac{\delta \mathcal{P}}{\delta e}\right) &= \frac{\delta^{\mathrm{T}}\mathcal{P}}{\delta e}\Lambda\frac{\partial}{\partial t}\left[M_1\frac{\partial e}{\partial z}+M_0(z)e\right]\\ &= \frac{\partial^{\mathrm{T}} e}{\partial t}M_0(z)\Lambda Q(e,z)\frac{\partial e}{\partial t}+\\ &\quad+\frac{\partial^{\mathrm{T}} e}{\partial t}M_1\Lambda Q(e,z)\frac{\partial}{\partial z}\frac{\partial e}{\partial t} \quad (30)\end{aligned}$$

Since $M_1\Lambda Q(e,z)$ is supposed symmetric, we can write that

$$\begin{aligned}\frac{\partial^{\mathrm{T}} e}{\partial t}M_1\Lambda Q(e,z)\frac{\partial}{\partial z}\frac{\partial e}{\partial t} &= \frac{\partial}{\partial z}\left[\frac{1}{2}\frac{\partial^{\mathrm{T}} e}{\partial t}M_1\Lambda Q(e,z)\frac{\partial e}{\partial t}\right]+\\ &\quad+\frac{\partial^{\mathrm{T}} e}{\partial t}M_1\Lambda\frac{\partial Q(e,z)}{\partial z}\frac{\partial e}{\partial t} \quad (31)\end{aligned}$$

If we substitute (31) in (30) and integrate on $[a,\, b]$, the first term in (31) depends only on the values of the co-energy variables at the boundary of the spatial domain, while the other terms, together with the contribution related to $\lambda\dot{\mathcal{P}}$ define $\frac{\partial^{\mathrm{T}} e}{\partial t}Q_a\frac{\partial e}{\partial t}$, with Q_a given in (29). ∎

Remark 4.1: From (22), (30) and (31), we deduce that $\dot{\mathcal{P}}_a$ is equal to the sum of two main contributions. The first one is a quadratic term in the time derivative of the co-energy variables, i.e. of $\frac{\partial e}{\partial t}$. The second one, instead, is a quantity that depends on the restriction of the co-energy variables and of their time derivative at the boundary of the domain, i.e., because of (19), on the boundary inputs and outputs, and their time derivative. In fact, if with some abuse in notation we define

$$\begin{aligned}M_\Lambda(u,y) &:= \begin{pmatrix}W\\ \tilde{W}\end{pmatrix}^{-\mathrm{T}} R^{-\mathrm{T}}\times\\ &\times\begin{pmatrix}M_1\Lambda Q(e(b),b) & 0\\ 0 & -M_1\Lambda Q(e(a),a)\end{pmatrix}R^{-1}\begin{pmatrix}W\\ \tilde{W}\end{pmatrix}^{-1}\end{aligned}$$

then after simple computations we have that the quantity related to the boundary conditions is

$$\frac{1}{2}\begin{pmatrix}\dot{u}\\ \dot{y}\end{pmatrix}^{\mathrm{T}}\left[\lambda\left(-M_W+M'_\partial\right)\begin{pmatrix}u\\ y\end{pmatrix}+M_\Lambda(u,y)\begin{pmatrix}\dot{u}\\ \dot{y}\end{pmatrix}\right] \quad (32)$$

If the symmetric part of Q_a obtained in (30) is negative semi-definite, (32) could be a good starting point for the definition of the boundary control action u that stabilises the system. This point is discussed in the remaining part of this section.

Remark 4.2: Remark 3.2 points out that the control design could benefit from the addiction to (28) of a function $\Psi_a(e(a),e(b),\dot{e}(a),\dot{e}(b),\dots)$ that depends on the co-energy variables, and their time derivatives, evaluated at the boundary of (1). In fact, this would lead to an admissible mixed-potential $\mathcal{P}_a$ which satisfies a novel balancing relation in which Ψ_a is a degree of freedom that allows to obtain different boundary control laws.

Now, let us consider the system (1) and denote by $y_\star$ the output associated with the equilibrium $x_\star(z)$ defined in (27). This means that, from (8), we have

$$\begin{pmatrix}u_\star\\ y_\star\end{pmatrix}=\begin{pmatrix}W\\ \tilde{W}\end{pmatrix}R\begin{pmatrix}\frac{\partial H}{\partial x}(x_\star(b))\\ \frac{\partial H}{\partial x}(x_\star(a))\end{pmatrix}\equiv\begin{pmatrix}W\\ \tilde{W}\end{pmatrix}R\begin{pmatrix}e_\star(b)\\ e_\star(a)\end{pmatrix} \quad (33)$$

being $(u_\star, y_\star)$ the input and the output associated with the equilibrium $x_\star(z)$, or $e_\star(z)$. The idea is to determine under which conditions the closed-loop system resulting from the feedback interconnection

$$u(t)=y_c(t) \qquad u_c(t)=y_\star-y(t) \quad (34)$$

of (1) with the (linear) control system

$$\begin{cases}\dot{x}_c(t)=A_cx_c(t)+B_cu_c(t)\\ y_c(t)=C_cx_c(t)+D_cu_c(t)\end{cases},\quad x_c\in\mathbb{R}^{n_c} \quad (35)$$

is asymptotically stable. Without loss of generality, let us assume that $u_\star=y_\star=0$, which implies that $e_\star(z)=0$.

The stabilising boundary controller (35) is determined thanks to a standard Lyapunov method by reling on the properties of a candiate Lyapunov functional $\mathcal{P}_{cl}$ associated with the closed-loop system, defined later on. If the requirements on existence and regularity of the trajectories (i.e., pre-compactness of the orbits) for the closed-loop system are met, then asymptotic stability can be proved on the basis of the extension to distributed parameter systems of the La Salle's Invariance Principle, [29, Theorem 3.64]. The main assumption is about the pre-compactness of the orbits. When H not quadratic (see Remark 2.2), checking this property is a difficult problem by itself, and its solution in the general case is beyond the scopes of this paper. However, for linear port-Hamiltonian systems, provided that the control action is generated by a passive system, the LaSalle's Invariance Principle can always be applied, see e.g. [30].

Starting from Proposition 4.1 and Remark 4.2 and with some abuse in notation, let us define a *shaped* closed-loop mixed-potential $\mathcal{P}_{cl}$ as:

$$\mathcal{P}_{cl}\left(e,x_c\right):=\mathcal{P}_a\left(e\right)+\Psi_a(u,y)+P_c(x_c)+\Psi_c(u_c,y_c) \quad (36)$$

where $\mathcal{P}_a$ is given in (28), and for simplicity

$$\begin{aligned}P_c(x_c)&=\frac{1}{2}x_c^{\mathrm{T}}\Gamma_{c0}x_c+\frac{1}{2}\dot{x}_c^{\mathrm{T}}\Gamma_{c1}\dot{x}_c\\ \Psi_c(u_c,y_c)&=\frac{1}{2}\begin{pmatrix}u_c\\ y_c\end{pmatrix}^{\mathrm{T}}\Gamma_c\begin{pmatrix}u_c\\ y_c\end{pmatrix}\\ \Psi_a(u,y)&=\frac{1}{2}\begin{pmatrix}u\\ y\end{pmatrix}^{\mathrm{T}}\Gamma_a\begin{pmatrix}u\\ y\end{pmatrix}\end{aligned} \quad (37)$$

Note that u, y, u_c and y_c are in fact function of e and x_c because of (6), (8), (34), and (35). In (37), it is assumed that Γ_{c0}, Γ_{c1}, Γ_c and Γ_a are symmetric and positive semi-definite.

To compute the variation of $\mathcal{P}_{cl}$ along system trajectories $(e(t),x_c(t))$, note that $\dot{\mathcal{P}}_a$ easily follows from Proposition 4.1

and Remark 4.1, while

$$\begin{aligned}\dot{P}_c(x_c) =& x_c^{\mathrm{T}}\Gamma_{c0}\left(A_c x_c + B_c u_c\right) + \\ &+ \dot{x}_c^{\mathrm{T}}\Gamma_{c1}\left(A_c\dot{x}_c + B_c\dot{u}_c\right)\\ \dot{\Psi}_c(u_c, y_c) =& \begin{pmatrix} u_c \\ C_c x_c + D_c u_c\end{pmatrix}^{\mathrm{T}}\Gamma_c \times \\ &\times \begin{pmatrix} \dot{u}_c \\ C_c\left(A_c x_c + B_c u_c\right)\end{pmatrix}\\ \dot{\Psi}_a(u, y) =& \begin{pmatrix} u \\ y\end{pmatrix}^{\mathrm{T}}\Gamma_a\begin{pmatrix}\dot{u}\\ \dot{y}\end{pmatrix}\\ =& \begin{pmatrix} -C_c x_c - D_c u_c \\ u_c\end{pmatrix}^{\mathrm{T}}\Gamma_a \times \\ &\times \begin{pmatrix} -C_c\left(A_c x_c + B_c u_c\right) - D_c\dot{u}_c \\ \dot{u}_c\end{pmatrix}\end{aligned} \tag{38}$$

where, for the last relation, (34) has been taken into account. This implies that $\dot{\mathcal{P}}_{cl}$ is equal to the sum of two contributions. From Proposition 4.1, we get that the first one is $\int_a^b \frac{\partial^{\mathrm{T}} e}{\partial t} Q_a \frac{\partial e}{\partial t}\mathrm{d}z$ in which $Q_a + Q_a^{\mathrm{T}} \leq 0$, while the second one is a quadratic function of $(x_c, \dot{x}_c, u_c, \dot{u}_c)$. These considerations lead to the next stability result, that is an immediate consequence of the La Salle's Invariance Principle in infinite dimensions, see e.g. [29, Theorem 3.64].

Proposition 4.2: Denote by $x_\star(z) \in L^2(a, b; \mathbb{R}^n)$ an equilibrium for (1), which corresponds to an $e_\star(z) \in H^1(a, b; \mathbb{R}^n)$ because of (27). Assume, without loss of generality, that $e_\star(z) = 0$, which implies that $y_\star = 0$, and that the (linear) control system (35) is interconnected to (1) according to (34). If there exists $\epsilon > 0$ such that along system trajectories

$$\begin{aligned}\dot{\mathcal{P}}_{cl}(e(t), x_c(t)) \leq \int_a^b \frac{\partial^{\mathrm{T}} e}{\partial t}(e(t,z)) Q_a(e(t,z))\cdot \\ \cdot \frac{\partial e}{\partial t}(e(t,z))\,\mathrm{d}z - \epsilon\,\dot{x}_c^{\mathrm{T}}(t)\dot{x_c}(t)\end{aligned} \tag{39}$$

with $Q_a + Q_a^{\mathrm{T}} \leq 0$, and if the largest invariant subset of

$$\begin{aligned}\Big\{(e(t), x_c(t)) \in H^1(a, b; \mathbb{R}^n) \times \mathbb{R}^{n_c} \mid \\ \dot{\mathcal{P}}_{cl}(e(t), x_c(t)) = 0\Big\}\end{aligned} \tag{40}$$

equals $\{(0, 0)\}$, then, under the assumptions of existence of solutions, and of pre-compactness of the orbits for the closed-loop system, the equilibrium point $e_\star(z) = 0$ is asymptotically stable.

Remark 4.3: The applicability of the La Salles Invariance Principle does not require that the Lyapunov functional, $\mathcal{P}_{cl}$ in this case, has a minimum at the equilibrium, but only that such functional is not increasing along system trajectories. This properties is guaranteed by (39). A good starting point in the control synthesis is to define $\mathcal{P}_{cl}$ thanks to (36) in such a way that it is radially unbounded and with, at least, a local minimum at the equilibrium. Then, provided that the closed-loop system is well-posed, i.e. that a solution exists, and that the orbits are pre-compact, the final step consists in checking the invariance properties of the orbits contained in the set (40) that defines the steady state.

Remark 4.4: In (39), the existence of such an $\epsilon > 0$ implies that in steady state the control system (35) brings system (1) to a configuration in which the boundary inputs and outputs u and y are constant. Moreover, from the first relation in (38), it is easy to see that if (39) holds we have that $\Gamma_{c1}A_c + A_c^{\mathrm{T}}\Gamma_{c1} \leq -2\epsilon I$, which means that A_c is Hurwitz, i.e. the control system (35) is asymptotically stable.

V. Example: the longitudinal vibrations of a beam

In this section, it is shown how the previous techniques can be applied to stabilise the longitudinal axial vibrations in a beam. Let us assume that the length of the bar is $L > 0$, and denote by $z \in [0,\, L]$ the spatial coordinate. As discussed in [14], $\varphi(t, z)$ denotes the longitudinal displacement of a section of the beam from the unstressed configuration, and $v(t, z) = \frac{\partial \varphi}{\partial t}(t, z)$ its velocity. The energy variables are the linear momentum density $p(t, z) = \rho S(z) v(t, z)$, where ρ is the material density and $S(z)$ the section of the beam, and the deformation $\varepsilon(t, z) = \frac{\partial \varphi}{\partial z}(t, z)$. Then, if the elastic behaviour is linear, the total energy is:

$$\mathcal{H}(p, \varepsilon) = \int_0^L \underbrace{\frac{1}{2}\left[\frac{p^2(z)}{\rho S(z)} + E S(z)\varepsilon^2(z)\right]}_{=: H(p,\varepsilon,z)}\mathrm{d}z \tag{41}$$

being E the Young elasticity modulus. This leads to the definition of the co-energy variables, namely the elastic force acting on the cross section $\sigma(t, z) = \frac{\partial H}{\partial \varepsilon}(\varepsilon(t, z)) = E S(z)\varepsilon(t, z)$, and its velocity $v(t, z) = \frac{\partial H}{\partial p}(p(t, z)) = \frac{p(t,z)}{\rho S(z)}$. The port-Hamiltonian formulation of the system is then

$$\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon(t,z)\\ p(t,z)\end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & -D\end{pmatrix}\begin{pmatrix} E S(z)\varepsilon(t,z) \\ \frac{1}{\rho S(z)} p(t,z)\end{pmatrix} \tag{42}$$

which is clearly in the form (1), being $D > 0$ the internal friction coefficient.

The boundary input and output are selected as

$$u(t) = \begin{pmatrix} v(t,0) \\ \sigma(t,L)\end{pmatrix} \quad y(t) = \begin{pmatrix} -\sigma(t,0) \\ v(t,L)\end{pmatrix} \tag{43}$$

which leads to the energy balance

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}(\varepsilon(t), p(t)) = -\int_0^L D\left[\frac{p(t,z)}{\rho S(z)}\right]^2 \mathrm{d}z + y^{\mathrm{T}}(t)u(t)$$

The system (1) is then in impedance form, since the supplied power through the boundary is equal to $y^{\mathrm{T}}u$.

From Corollary 3.1, we have that (42) admits the Brayton–Moser formulation

$$\underbrace{\begin{pmatrix} -\frac{1}{E S(z)} & 0 \\ 0 & \rho S(z)\end{pmatrix}}_{=: Q(z)}\frac{\partial}{\partial t}\begin{pmatrix}\sigma\\ v\end{pmatrix} = \begin{pmatrix}\frac{\delta \mathcal{P}}{\delta \sigma}(\sigma, v) \\ \frac{\delta \mathcal{P}}{\delta v}(\sigma, v)\end{pmatrix} = \begin{pmatrix} -\frac{\partial v}{\partial z} \\ \frac{\partial \sigma}{\partial z} - D v\end{pmatrix} \tag{44}$$

with mixed-potential

$$\mathcal{P}(\sigma, v) = \frac{1}{2}\int_0^L \left(v\frac{\partial \sigma}{\partial z} - \sigma\frac{\partial v}{\partial z} - Dv^2 \right) \mathrm{d}z \tag{45}$$

for which the following balance relation holds true:

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = \int_0^L \left[-\frac{1}{ES(z)}\left(\frac{\partial \sigma}{\partial t}\right)^2 + \rho S(z)\left(\frac{\partial v}{\partial t}\right)^2 \right] \mathrm{d}z + \\ + \frac{1}{2}\left[\frac{\partial \sigma}{\partial t}(z)v(z) - \sigma(z)\frac{\partial v}{\partial t}(z) \right]_0^L \tag{46}$$

Since the system is linear, without loss of generality, we assume that the equilibrium is $(\sigma_\star, v_\star) = (0, 0)$. From (44) and (46), we see that $\mathcal{P}$ and Q cannot be used *as is* for control design or to prove stability of equilibria. So, with Proposition 4.1 in mind, the first step is to determine $\mathcal{P}_a$ and Q_a to be used in the Lyapunov analysis. Furthermore, as discussed in Remark 4.3, to facilitate the control synthesis, the idea is to have also $\mathcal{P}_a$ with a (local) minimum in $(0, 0)$. At first, note that $\mathcal{P}$ in (45) can be re-written as

$$\mathcal{P}(\sigma, v) = \int_0^L \left[-\frac{1}{2D}\left(\frac{\partial \sigma}{\partial z} - Dv\right)^2 + \right. \\ \left. + \frac{1}{2D}\left(\frac{\partial \sigma}{\partial z}\right)^2 \right] \mathrm{d}z - \frac{1}{2}\left[\sigma(z)v(z)\right]_0^L \tag{47}$$

since $D > 0$. So, if in Proposition 4.1 we take $\lambda > 0$ and $\Lambda(z) = \begin{pmatrix} ES(z) & 0 \\ 0 & \frac{1}{\rho S(z)} \end{pmatrix}$, we have that the term under the integral in $\mathcal{P}_a$ defined as in (28) becomes

$$ES(z)\left(\frac{\partial v}{\partial z}\right)^2 + \frac{\lambda}{2D}\left(\frac{\partial \sigma}{\partial z}\right)^2 + \\ + \left(\frac{1}{\rho S(z)} - \frac{\lambda}{2D}\right)\left(\frac{\partial \sigma}{\partial z} - Dv\right)^2$$

which is always greater than 0 if $\lambda < \frac{2D}{\rho S(z)}$, for all $z \in [0,\, L]$, and positive, [20]. This property holds also for $\mathcal{P}_a$ once the term that depends on the boundary conditions (or on u and y) that appears in (47) is properly cancelled. With Remark 4.2 in mind, in (36) we define $\Psi_a(\sigma, v) = \frac{\lambda}{2}\left[\sigma(z)v(z)\right]_0^L$: such expression can be equivalently re-written in terms of u and y because of (43).

The control system (35) is designed in order to render the closed-loop system asymptotically stable. Let us now assume that $x_c = (x_{c0}, x_{cL}) \in \mathbb{R}^2$. The controller dynamics and the functions P_c and Ψ_c in (36) have to be selected so that $\mathcal{P}_{cl}$ meets the conditions of Proposition 4.2. A possible choice is the following:

$$\Psi_c(\sigma, v) = \lambda\left[\sigma(0)v(0) + \frac{g_L}{2}\sigma^2(L) - \frac{d_0}{2}v^2(0)\right] \\ = -\frac{\lambda}{2d_0}\left[\sigma(0) + d_0 v(0)\right]^2 + \frac{\lambda}{2d_0}\sigma^2(0) + \\ + \frac{\lambda}{2}g_L\sigma^2(L)$$

$$P_c(x_{c0}, x_{CL}) = \frac{1}{2}\left[\gamma_0 \dot{x}_{c0}^2 + \gamma_L \dot{x}_{cL}^2\right]$$

where g_L, d_0, γ_0 and γ_L are positive constants. If

$$\begin{cases} \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_L \end{pmatrix}\begin{pmatrix} \dot{x}_{c0} \\ \dot{x}_{cL} \end{pmatrix} = \begin{pmatrix} -d_0 & 0 \\ 0 & -g_L \end{pmatrix}\begin{pmatrix} x_{c0} \\ x_{cL} \end{pmatrix} + u_c \\ y_c = \begin{pmatrix} x_{c0} \\ x_{cL} \end{pmatrix} \end{cases} \tag{48}$$

then from (34) with $y_\star = 0$, i.e. $u = -y_c$ and $u_c = y$, it is possible to verify that $\mathcal{P}_{cl}$ is positive and with a minimum in the origin if $\lambda < \min\left(\frac{2D}{\rho S(z)}, \frac{2d_0}{\gamma_0}\right)$, for $z \in [0,\, L]$, and positive. The control system (48) is passive and, from a physical point of view, it is equivalent to the interconnection of a mass and a damper in series in $z = 0$, and to the parallel of a spring and a damper in $z = L$ since $x_{c0} = -v(0)$ and $x_{cL} = -\sigma(L)$. Passivity of (48) implies that the closed-loop system is well-posed [13], [30], and that it makes sense to evaluate $\mathcal{P}_{cl}$ along system trajectories. In particular, from (36) and after some computations, we get that

$$\frac{\mathrm{d}\mathcal{P}_{cl}}{\mathrm{d}t} = \int_0^L \left\{ -\frac{\lambda}{ES(z)}\left(\frac{\partial \sigma}{\partial t}\right)^2 - \right. \\ \left. - \left[D - \lambda\rho S(z)\right]\left(\frac{\partial v}{\partial t}\right)^2 \right\} \mathrm{d}z - \\ - (d_0 - \lambda\gamma_0)\dot{x}_{c0}^2 - (g_L + \lambda\gamma_L)\dot{x}_{cL}^2 \leq 0$$

with now $0 < \lambda < \min\left(\frac{D}{\rho S(z)}, \frac{d_0}{\gamma_0}\right)$, $z \in [0,\, L]$.

The functional $\mathcal{P}_{cl}$ and the closed-loop system satisfy the conditions of Proposition 4.2. More precisely, as discussed in [30], [31], the closed-loop dynamics can be written on an extended state space in the form $\dot{\xi} = \mathcal{J}_e\xi$, where $\xi = (\sigma, v, x_{c0}, x_{cL})$ and $\mathcal{J}_e$ is a linear operator. Under standard assumptions on the state space and on the domain of $\mathcal{J}_e$, it is possible to prove that $\mathcal{J}_e$ generates a contraction semigroup. Moreover, due to the fact that $(\lambda_e I - \mathcal{J}_e)^{-1}$ is compact for some $\lambda_e > 0$, the orbits are compact, and the trajectories converge to the largest invariant set contained in $\dot{\mathcal{P}}_{cl} = 0$, i.e. to the set (40). From the previous balance relation and from the analisys of the steady state configuration of the closed-loop system, it is possible to check that such invariant contains only the origin, which proves that the equilibrium is asymptotically stable

VI. Conclusions and future work

In this paper, for a given class of distributed port-Hamiltonian systems with one dimensional spatial domain, an equivalent Brayton–Moser formulation is presented. The result is a gradient equation with respect to a mixed-potential function that has the dimensions of power. Differently from port-Hamiltonian systems in which the Hamiltonian is a good Lyapunov function candidate, this mixed-potential is in general not bounded from below and not decreasing along system trajectories e.g. when the input is set equal to zero. Then, to use such quantity for the synthesis of boundary control laws, a procedure to generate a family of mixed-potential functions suitable for the Lyapunov analysis is presented. This is the starting point for computing the control

action that stabilises a desired equilibrium. The procedure has been presented with the help of a simple example, i.e. the stabilisation of longitudinal vibrations in an elastic beam. Future work deals with the extension of the proposed Brayton–Moser formulation to a larger class of systems, and in determining a more systematic procedure for the control synthesis based on this new framework as did e.g in [13]–[15] for distributed port-Hamiltonian systems within the energy-shaping plus damping injection paradigm.

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