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Dynamic Output Feedback Invariants of Full Relative Degree Nonlinear SISO Systems

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Abstract— **The goal of this paper is to explicitly describe invariants of a plant described by a Chen–Fliess series under a class of dynamic output feedback laws using earlier work by the authors on feedback transformation groups. The main result requires the rather strong assumption that the plant has a generating series with both finite Lie rank and full relative degree. In which case, there is no loss of generality in working with state space realizations of the plant. An additional genericness assumption regarding the normal form of the plant is also required, but as shown by the examples, this condition is often available in typical problems. All of the analysis presented is restricted to the single-input, single-output case.**

Index Terms— **Nonlinear systems, dynamic output feedback, Chen–Fliess series**

AMS Subject Classifications— **93C10, 93B52, 93B25**

I. Introduction

The study of feedback invariants has a long history in control theory beginning with the work of Brockett, Hammer and Krishnaprasad in the case of linear systems [3], [4], [14] and Brockett, Jakubczyk, Respondek, and many others in the context of nonlinear state space systems [2], [16], [17], [20]. More recently in [7], [11], the authors have been interested in identifying invariants under dynamic output feedback of nonlinear systems described in terms of Chen–Fliess functional expansions or Fliess operators [6]. The basic idea was to employ various combinatorial algebras of formal power series induced by system interconnections [8] to identify a transformation group that describes the action of such feedback on a given system. The approach has been incremental in that first a general *output feedback transformation group* was developed with a corresponding right action which describes the result of applying such feedback to *any* possible plant. Then an algorithm was given to identify a class of invariant subseries contained by the generating series of a plant. Various conditions were presented under which this invariant series was *maximal*, but in general the results given by this algorithm are known to be conservative. Now in [11] a somewhat implicit description was given of an output feedback invariant series given a *specific plant* and an arbitrary Fliess operator in the feedback path. This is the starting point for the present work, where the goal is to provide a completely *explicit* description of such an invariant series under certain assumptions.

The first step towards this goal is to simply describe the relevant subgroup of the output feedback transformation group. Then the main result describes under what conditions the algorithm given in [7] applies to the problem at hand. This will require the rather strong assumption that the plant has a generating series with both finite Lie rank and full relative degree. In which case, there is no loss of generality in working with state space realizations of the plant. An additional genericness assumption regarding the normal form of the plant is also required, but as shown by the examples, this condition is often available in typical problems. All of the analysis presented is restricted to the single-input, single-output case. Finally, it should be stated that a much larger feedback transformation group has also been studied by one of the authors in [10], but most of the results presented there do not provide any interesting specificity here.

The paper is organized as follows. First some preliminaries are introduced for readers not familiar with Fliess operators and their associated interconnection algebras. Then a short summary of dynamic output feedback invariants for linear time-invariant (LTI) systems is given. This provides some motivation and a basis for comparison against the nonlinear case. The main results are presented in Section IV and then applied to four examples in the subsequent section. The main conclusion and some suggestions for future work are given in the final section.

II. Preliminaries

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. Let $|\eta|_{x_i}$ denote the number of times the letter $x_i \in X$ appears in the word η. The set of all words including the empty word, $\emptyset$, is designated by X^*, and $X^+ := X^* - \{\emptyset\}$. The set X^* forms a monoid under catenation. The set ηX^* is comprised of all words with the prefix η. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. If $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X \rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product defined in terms of the shuffle product of two words

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$ [6].

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A. Fliess Operators

One can formally associate with any series $c \in \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : \ 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_{\mathfrak{p}}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B^m_{\mathfrak{p}}(R)[t_0, t_1] := \{u \in L^m_{\mathfrak{p}}[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0)$$

[6]. If there exists constants $K, M > 0$ such that

$$|(c, \eta)| \leq K M^{|\eta|}\, |\eta|!, \quad \forall \eta \in X^*,$$

then F_c constitutes a well defined mapping from $B^m_{\mathfrak{p}}(R)[t_0,\, t_0{+}T]$ into $B^{\ell}_{\mathfrak{q}}(S)[t_0,\, t_0{+}T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p}{+}1/\mathfrak{q} = 1$ [12]. The set of all such *locally convergent* generating series is denoted by $\mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$. A Fliess operator F_c is *realizable* on a ball $B^m_{\mathfrak{p}}(R)[t_0, t_0 + T]$ whenever there exists a system of n ordinary differential equations and a set of ℓ output functions,

$$\dot{z}(t) = g_0(z(t)) + \sum_{i=1}^{m} g_i(z(t))\, u_i(t), \quad z(t_0) = z_0 \tag{1a}$$

$$y_j(t) = h_j(z(t)), \quad j = 1, \ldots, \ell, \tag{1b}$$

where each analytic vector field g_i is written in terms of local coordinates on a neighborhood $\mathcal{W}$ of z_0, every real-valued function h_j is analytic on $\mathcal{W}$, and (1a) has a solution z well defined on $[t_0, t_0 + T]$ for any input $u \in B^m_{\mathfrak{p}}(R)[t_0, t_0 + T]$ satisfying $y_j(t) = F_{c_j}[u](t) = h_j(z(t))$, $j = 1, \ldots, \ell$. It is easily verified that for any $\eta = x_{i_k} \cdots x_{i_1} \in X^*$

$$(c_j, \eta) = L_{g_\eta} h_j(z_0) := L_{g_{i_1}} \cdots L_{g_{i_k}} h_j(z_0),$$

where $L_{g_i} h_j$ denotes the *Lie derivative* of the output h_j with respect to the vector field g_i. A given operator F_c can be shown to be realizable if and only if its generating series $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ has finite Lie rank [6], [15].

B. Algebras Induced by System Interconnections

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \,\sqcup\!\sqcup\, d}$, respectively [6]. When Fliess operators F_c and F_d with $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X \rangle\rangle$ are interconnected in a cascade fashion, the

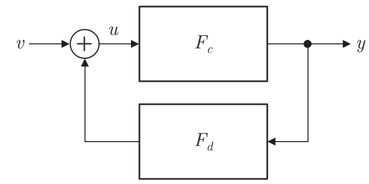

Fig. 1. Feedback interconnection of Fliess operators F_c and F_d

composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c \circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \psi_d(\eta)(\mathbf{1}) \tag{2}$$

[5]. Here $\mathbf{1}$ denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X \rangle\rangle$ to the set of vector space endomorphisms on $\mathbb{R}\langle\langle X \rangle\rangle$, $\mathrm{End}(\mathbb{R}\langle\langle X \rangle\rangle)$, uniquely specified by $\psi_d(x_i \eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with $\psi_d(x_i)(e) = x_0(d_i \,\sqcup\!\sqcup\, e)$, $i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X \rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := 1$). By definition $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$.

When two Fliess operators F_c and F_d are interconnected to form a feedback system as shown in Figure 1, the generating series for the closed-loop system is denoted by the feedback product $c@d$. It can be computed explicitly using the Hopf algebra of coordinate functions associated with the underlying *output feedback group* [8]. Specifically, in the SISO case, where $X = \{x_0, x_1\}$ and $m = \ell = 1$, define the set of *unital* Fliess operators $\mathscr{F}_\delta = \{I + F_c \ : \ c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle\}$, where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta + c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}_{LC}\langle\langle X_\delta \rangle\rangle$. The central idea is that $(\mathscr{F}_\delta, \circ, I)$ forms a group of operators under the composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + c \circledcirc d$, $c \circledcirc d := d + c \,\tilde{\circ}\, d_\delta$, and $\tilde{\circ}$ denotes the *mixed* composition product. That is, the product

$$c \,\tilde{\circ}\, d_\delta = \sum_{\eta \in X^*} (c, \eta)\, \phi_d(\eta)(\mathbf{1}),$$

where ϕ_d is analogous to ψ_d in (2) except here $\phi_d(x_i)(e) = x_i e + x_0(d_i \,\sqcup\!\sqcup\, e)$ with $d_0 := 0$. The coordinate maps for the corresponding Hopf algebra H have the form

$$a_\eta : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R} : c \mapsto (c, \eta),$$

where $\eta \in X^*$. The commutative product is defined as

$$\mu : a_\eta \otimes a_\xi \mapsto a_\eta a_\xi,$$

where the unit $\mathbf{1}$ is defined to map every c to zero. If the *degree* of a_η is defined as $\deg(a_\eta) = 2\,|\eta|_{x_0} + |\eta|_{x_1} + 1$, then H is graded and connected with $H = \bigoplus_{n \geq 0} H_n$, where H_n is the set of all elements of degree n and $H_0 = \mathbb{R}\mathbf{1}$.

The coproduct Δ is defined so that the formal power series product $c \circledcirc d$ for the group $\mathscr{F}_\delta$ satisfies

$$\Delta a_\eta(c,d) = a_\eta(c \circledcirc d) = (c \circledcirc d, \eta).$$

Of primary importance is the following lemma which describes how the group inverse $c_\delta^{-1} := \delta + c^{-1}$ is computed.

Lemma 1: [8] The Hopf algebra (H, μ, Δ) has an antipode S satisfying $a_\eta(c^{-1}) = (Sa_\eta)(c)$ for all $\eta \in X^*$ and $c \in \mathbb{R}\langle\langle X \rangle\rangle$.

With this concept, the generating series for the feedback connection, $c@d$, can be computed explicitly.

Theorem 1: [8] For any $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ it follows that

$$c@d = c \,\tilde{\circ}\, (-d \circ c)_\delta^{-1}.$$

It is shown in [7] that the feedback product can be viewed in terms of the group $(\mathbb{R}\langle\langle X_\delta \rangle\rangle, \circ, \delta)$ acting as a right transformation group on the set $\mathbb{R}\langle\langle X \rangle\rangle$ via the mixed composition product.[1] This group contains all possible dynamic output feedback laws for all possible plants, but it also contains a bit more since, for example, $\delta + x_1 \in \mathbb{R}\langle\langle X_\delta \rangle\rangle$ even though there does not exist $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ such that $x_1 = -d \circ c$. Therefore, a series being invariant under the right transformation group $(\mathbb{R}\langle\langle X_\delta \rangle\rangle, \circ, \delta)$ is a sufficient condition for being invariant under dynamic output feedback. Of particular importance is the following invariance theorem, which uses the fact that every $c \in \mathbb{R}\langle\langle X \rangle\rangle$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k \geq 0} (c, x_0^k) x_0^k$ and $c_F := c - c_N$.

Theorem 2: [7] Every nonzero series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ which is not equivalent to c_N can be decomposed into the form $c = c_i + \tilde{c}$, where $\mathrm{supp}(c_i) \cap \mathrm{supp}(\tilde{c}) = \emptyset$,

$$c_i = c_{N_0} + x_0^{r_1 - 1} x_1 c_{N_1} + x_0^{r_1 - 1} x_1 x_0^{r_2 - 1} x_1 c_{N_2} + \cdots,$$

$c_{N_\ell} \in \mathbb{R}[X_0]$ (a polynomial in x_0), $r_\ell \geq 1$, $\deg(c_{N_\ell}) \leq r_{\ell+1} - 1$, and c_i is a nonzero invariant series under the right transformation group $(\mathbb{R}\langle\langle X_\delta \rangle\rangle, \circ, \delta)$. That is, $c \,\tilde{\circ}\, e_\delta = c_i + c_v(e)$ with $\mathrm{supp}(c_i) \cap \mathrm{supp}(c_v(e)) = \emptyset$ for all $e \in \mathbb{R}\langle\langle X \rangle\rangle$.

The proof of the theorem is constructive. It boils down to identifying pairs $(c_{N_\ell}, r_{\ell+1})$, $\ell \geq 0$ to extract the invariant c_i from c via the following algorithm:

Step 1: Set $\ell = 0$.

Step 2: Write c in the canonical form

$$c = c_N + x_0^{r-1} c_1 + x_0^r c_2 + \cdots,$$

where $r \geq 1$, c_k are proper series with $x_0^{-1}(c_k) = 0$ for all $k \geq 1$, and $c_1 \neq 0$.

Step 3: Define $c_{N_\ell} = \sum_{k=0}^{r-1} (c_N, x_0^k) x_0^k$ and $r_{\ell+1} = r$.

Step 4: Redefine $c = x_1^{-1}(c_1)$ and set $\ell = \ell + 1$.

Step 5: If $|c|_{x_1} = 0$ set $c_{N_\ell} = c$, $c_{N_k} = 0$, $k > \ell$ and stop. Otherwise, return to Step 2.

The algorithm will only terminate when c is *input-limited*, that is, when $\max_{\eta \in \mathrm{supp}(c)} |\eta|_{x_1}$ is finite. There is no claim that the series c_i is *maximal* in the sense that its support contains the support of any other series which is also invariant under $\mathbb{R}\langle\langle X_\delta \rangle\rangle$. On the other hand, as will be addressed later in the paper, it may be too conservative if c is fixed (i.e, the plant is given) and only group elements of the form $(-d \circ c)_\delta$, $d \in \mathbb{R}\langle\langle X \rangle\rangle$ are admissible.

C. Relative Degree

The standard definition of relative degree for any input-affine state space realization (1) of $y = F_c[u]$ with $c \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ is given in [15]. But the concept can also be defined solely in terms of formal power series concepts. The starting point is the notion of a *linear word* when $X = \{x_0, x_1\}$, namely, any word in the language $\mathscr{L} = \{\eta \in X^* : \eta = x_0^{n_1} x_1 x_0^{n_0},\ n_1, n_0 \geq 0\}$.

Definition 1: [9] Given $c \in \mathbb{R}\langle\langle X \rangle\rangle$, let $r \geq 1$ be the largest integer such that $\mathrm{supp}(c_F) \subseteq x_0^{r-1} X^*$. Then c has *relative degree* r if the linear word $x_0^{r-1} x_1 \in \mathrm{supp}(c)$, otherwise it is not well defined.

Observe that c having relative degree r is equivalent to saying that

$$c = c_N + c_F = c_N + K x_0^{r-1} x_1 + x_0^{r-1} e$$

for some $K \neq 0$ and some proper $e \in \mathbb{R}\langle\langle X \rangle\rangle$ with $x_1 \notin \mathrm{supp}(e)$. It is shown in [9], [10] that this definition is equivalent to the standard definition when F_c is realizable.

III. Dynamic Output Feedback Invariants for Linear Time-invariant Systems

Consider a SISO LTI system $u(t) \mapsto y(t)$ with an irreducible transfer function

$$\begin{aligned} \frac{y(s)}{u(s)} = h(s) &= K \frac{b_0 + b_1 s + \cdots + b_{n-r-1} s^{n-r-1} + s^{n-r}}{a_0 + a_1 s \cdots + \cdots a_{n-1} s^{n-1} + s^n} \\ &= K \frac{b(s)}{a(s)}, \end{aligned}$$

where $K \neq 0$ and $1 \leq r < n$. In a calculation originating with the work of Kalman [18] and carried out explicitly by Hammer [14], one can divide $b(s)$ into $a(s)$ so that $a(s) = b(s)p(s) + r(s)$ with $(r(s), b(s))$ being a coprime pair of polynomials

$$\begin{aligned} p(s) &= p_0 + p_1 s + \cdots + p_{r-1} s^{r-1} + s^r \\ r(s) &= r_0 + r_1 s + \cdots + r_{n-r-2} s^{n-r-2} + r_{n-r-1} s^{n-r-1} \end{aligned}$$

and $\deg(r(s)) < \deg(b(s))$.[2] In which case,

$$h(s) = \frac{K}{p(s) + \frac{r(s)}{b(s)}} = \frac{K}{p(s)} \left(1 + \frac{r(s)}{b(s)} \frac{1}{p(s)}\right)^{-1}, \qquad (3)$$

so that $h(s)$ can be viewed as the feedback interconnection shown in Figure 2. At least three interpretations of (3) are possible as described next: a factorization point of view, an interpretation in terms of state space realizations, and a power series perspective.

If dynamic output feedback $u(s) = v(s) - g(s)y(s)$ is applied for some strictly proper $g(s)$ then it is immediate from (3) that the closed-loop system is

$$\frac{y(s)}{v(s)} = h_{cl}(s) = \frac{K}{p(s) + \left[\frac{r(s)}{b(s)} + K g(s)\right]},$$

[1] The same composition symbol will be used for the group product on $\mathbb{R}\langle\langle X_\delta \rangle\rangle$ and series composition $c \circ d$ on $\mathbb{R}\langle\langle X \rangle\rangle$. The distinction between c and $c_\delta = \delta + c$ will always make it clear which product is at play.

[2] The relative degree r and the remainder $r(s)$ will always be distinguished by the argument '(s)'.

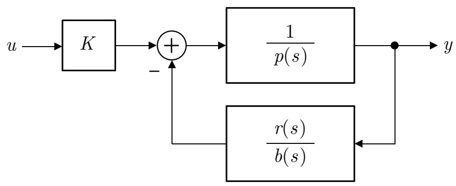

Fig. 2. LTI system $h(s)$ viewed as a feedback connection

implying that the polynomial $p(s)$ is invariant under this feedback class (henceforth, abbreviated as *dynamic output feedback*). In particular, it can happen that the all pole system $K/p(s)$ is not equal to K/s^r and thus does not have a Brunovsky type state space realization. In the special case where $h(s)$ is minimum phase and $1/p(s)$ is stable, equation (3) is a right coprime factorization $h(s) = Kn(s)d^{-1}(s)$ since the Bezout equation

$$\tilde{n}(s)\underbrace{\left[\frac{1}{p(s)}\right]}_{n(s)} + \tilde{d}(s)\underbrace{\left[1 + \frac{r(s)}{b(s)}\frac{1}{p(s)}\right]}_{d(s)} = 1$$

has the stable solution $\tilde{n}(s) = -r(s)/b(s)$ and $\tilde{d}(s) = 1$. Thus, the controller $Kg(s) = \tilde{d}^{-1}(s)\tilde{n}(s) = -r(s)/b(s)$ yields the stable closed-loop system $h_{cl}(s) = K/p(s)$ [13].

To derive a state space interpretation of (3), let (A_1, b_1, c_1) and (A_2, b_2, c_2) be minimal realizations of $1/p(s)$ and $r(s)/b(s)$, respectively. Then a realization of $h(s)$ follows directly from the feedback structure in (3) to be

$$\begin{aligned}
\dot{z} &= \left[\begin{array}{c|c} A_1 & -b_1c_2 \\ \hline b_2c_1 & A_2 \end{array}\right] z + \begin{bmatrix} Kb_1 \\ 0 \end{bmatrix} u, \quad z(0) = z_0 \\
y &= \left[\begin{array}{c|c} c_1 & 0 \end{array}\right] z.
\end{aligned}$$

If both realizations are assumed to be in controller canonical form, then this realization becomes

$$\dot{z}_1 = z_2 \tag{4a}$$
$$\dot{z}_2 = z_3 \tag{4b}$$
$$\vdots$$
$$\dot{z}_{r-1} = z_r \tag{4c}$$
$$\dot{z}_r = P\xi + R\eta + Ku \tag{4d}$$
$$\dot{\eta} = S\xi + Q\eta \tag{4e}$$
$$y = z_1, \tag{4f}$$

where $\xi = [z_1 \cdots z_r]$, $\eta = [z_{r+1} \cdots z_n]$, $P = -[p_0 \cdots p_{r-1}]$, $R = -[r_0 \cdots r_{n-r-1}]$, $S = e_{n-r}e_1^T(r)$, and

$$Q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-r-1} \end{bmatrix}.$$

(Here $e_i(j) \in \mathbb{R}^j$ has a one in the i-th position and zero elsewhere. The notation is simplified to e_i if $i = j$.) This is the *Brynes-Isidori normal form* which appears in the context of feedback linearization [15]. If $r < n-1$ then $\dot{\eta} = Q\eta$, $\eta(0) = \eta_0$ are the zero dynamics of the system. Setting $u = v - w$ in (4d), where

$$\begin{aligned}
\dot{\tilde{\eta}} &= Q\tilde{\eta} + e_{n-r}y, \quad \tilde{\eta}(0) = \eta_0 \\
w &= \frac{R}{K}\tilde{\eta},
\end{aligned}$$

it follows that $\tilde{\eta}(t) = \eta(t)$, $t \geq 0$ since both the zero dynamics and the feedback system $y \mapsto w$ are driven in precisely the same way by the output y. In which case, the closed-loop system becomes

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= P\xi + Kv \\
\dot{\eta} &= S\xi + Q\eta, \quad \eta(0) = \eta_0 \\
\dot{\tilde{\eta}} &= e_{n-r}y + Q\tilde{\eta}, \quad \tilde{\eta}(0) = \eta_0 \\
y &= z_1.
\end{aligned}$$

The dynamics of ξ are the *invariant dynamics* of the system under dynamic output feedback. As expected, $Kg(s) := Kw(s)/y(s) = -r(s)/b(s)$, and the output y is zero precisely when $\xi(0) = 0$ and $v(t) = 0$, $t \geq 0$ implying that $u(t) = u^*(t) := (R/K)\mathrm{e}^{Qt}\eta(0)$.

From a power series point of view, one can write (3) as

$$\begin{aligned}
h(s) &= \frac{K}{p(s)}\sum_{k=0}^{\infty}\left(-\frac{r(s)}{b(s)}\frac{1}{p(s)}\right)^k \\
&= \frac{K}{p(s)} - K\frac{r(s)}{b(s)}\frac{1}{p^2(s)} + \mathcal{O}\left(\left(\frac{r(s)}{b(s)}\right)^2\frac{1}{p^3(s)}\right).
\end{aligned}$$

The infinite series is well defined as a formal power series because the strict properness of $r(s)/(b(s)p(s))$ implies the series is locally finite, and hence summable [1]. It is clear that the first $r+1$ terms of this series expansion will always coincide with the first $r+1$ terms of

$$\frac{K}{p(s)} = h_r s^{-r} + h_{r+1}s^{-r-1} + \cdots + h_{2r}s^{-2r} + \mathcal{O}(s^{-2r-1}).$$

Furthermore, these specific terms are invariant under dynamic output feedback, and therefore the polynomial $p_h(x) := h_r x^r + h_{r+1}x^{r+1} + \cdots + h_{2r}x^{2r}$ is invariant in this sense. In fact, this polynomial is maximal in that there is no higher degree polynomial having this invariance property.

Example 1: Consider the case where $r = 3$ so that $K/p(s) = K/(p_0 + p_1 s + p_2 s^2 + s^3)$. A simple calculation gives $p_h(x) = h_3x^3 + h_4x^4 + h_5x^5 + h_6x^6$, where

$$h_3 = K \tag{5a}$$
$$h_4 = -Kp_2 \tag{5b}$$
$$h_5 = K(p_2^2 - p_1) \tag{5c}$$
$$h_6 = K(2p_1p_2 - p_2^3 - p_0). \tag{5d}$$

$\square$

IV. Dynamic Output Feedback Invariants of a Given Plant

In this section, the problem of determining a dynamic output feedback invariant series for a given plant is considered. It is shown first that the underlying transformation group for this problem is an additive subgroup of $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$.

Definition 2: For any fixed $c \in \mathbb{R}\langle\langle X\rangle\rangle$, define

$$\mathcal{O}_c = \{e_\delta \in \mathbb{R}\langle\langle X_\delta\rangle\rangle : e_\delta = (d \circ c)_\delta,\ d \in \mathbb{R}\langle\langle X\rangle\rangle\}.$$

Theorem 3: For any series $c \in \mathbb{R}\langle\langle X\rangle\rangle$, the triple $(\mathcal{O}_c, +, \delta)$ defines an additive group, where

$$e_\delta + e'_\delta = (d \circ c)_\delta + (d' \circ c)_\delta := ((d + d') \circ c)_\delta$$

for any $e_\delta = (d \circ c)_\delta, e'_\delta = (d' \circ c)_\delta \in \mathcal{O}_c$.

Proof: The claim follows directly from the left linearity of the composition product on the $\mathbb{R}$-vector space $\mathbb{R}\langle\langle X\rangle\rangle$. ■

Theorem 4: The additive group $(\mathbb{R}\langle\langle X\rangle\rangle, +, 0)$ acts on the set $\mathbb{R}\langle\langle X\rangle\rangle$ as a right transformation group, where the action is given by the output feedback product. That is, $c@0 = c$ and

$$(c@d_1)@d_2 = c@(d_1 + d_2).$$

Proof: The first identity is trivial. For the second, two algebraic facts are needed. First, as described in Lemma 1, the composition inverse is defined in terms of a Hopf algebra antipode, S, using the group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$. Such an S is always an antihomomorphism for both the algebra and the coalgebra structures on H, for example, $S(a_1a_2) = S(a_2)S(a_1)$, $\forall a_1, a_2 \in H$. Therefore, it follows directly that $(c_\delta \circ d_\delta)^{-1} = d_\delta^{-1} \circ c_\delta^{-1}$. Second, it is easily checked that the following associativity property holds

$$c \circ (d \,\tilde{\circ}\, e_\delta) = (c \circ d) \,\tilde{\circ}\, e_\delta$$

keeping in mind that

$$F_c(F_{d \,\tilde{\circ}\, e_\delta}) = F_c(F_d(I + F_e)) = F_{c\circ d}(I + F_e).$$

Proceeding with the calculation, it follows by definition of the output feedback product and the fact that $(\mathbb{R}\langle\langle X\rangle\rangle, \circ, \delta)$ is known to act as a right transformation on $\mathbb{R}\langle\langle X\rangle\rangle$ via the product $c \,\tilde{\circ}\, d_\delta$ that

$$\begin{aligned}
&(c@d_1)@d_2 \\
&\quad = (c \,\tilde{\circ}\, (-d_1 \circ c)_\delta^{-1})@d_2 \\
&\quad = (c \,\tilde{\circ}\, (-d_1 \circ c)_\delta^{-1}) \,\tilde{\circ}\, (-d_2 \circ (c \,\tilde{\circ}\, (-d_1 \circ c)_\delta^{-1}))_\delta^{-1} \\
&\quad = c \,\tilde{\circ}\, \left[(-d_1 \circ c)_\delta^{-1} \circ (-d_2 \circ (c \,\tilde{\circ}\, (-d_1 \circ c)_\delta^{-1}))_\delta^{-1}\right].
\end{aligned}$$

Now apply the first fact stated above, the definition of the group product on $\mathbb{R}\langle\langle X_\delta\rangle\rangle$, and the second fact in this order to get

$$\begin{aligned}
(c@d_1)@d_2 &= c \,\tilde{\circ}\, \left[(-d_2 \circ (c \,\tilde{\circ}\, (-d_1 \circ c)_\delta^{-1}))_\delta \circ \right. \\
&\qquad \left. (-d_1 \circ c)_\delta\right]^{-1} \\
&= c \,\tilde{\circ}\, \left[(-d_1 \circ c) + (-d_2 \circ (c \,\tilde{\circ}\, (-d_1 \circ c)_\delta^{-1})) \,\tilde{\circ}\, \right. \\
&\qquad \left. (-d_1 \circ c)_\delta\right]_\delta^{-1} \\
&= c \,\tilde{\circ}\, \left[(-d_1 \circ c) + ((-d_2 \circ c) \,\tilde{\circ}\, (-d_1 \circ c)_\delta^{-1}) \,\tilde{\circ}\, \right. \\
&\qquad \left. (-d_1 \circ c)_\delta\right]_\delta^{-1}.
\end{aligned}$$

Finally, just simplify the result using properties already stated so that

$$\begin{aligned}
(c@d_1)@d_2 &= c \,\tilde{\circ}\, \left[(-d_1 \circ c) + (-d_2 \circ c) \,\tilde{\circ}\, ((-d_1 \circ c)_\delta^{-1} \circ \right. \\
&\qquad \left. (-d_1 \circ c)_\delta)\right]_\delta^{-1} \\
&= c \,\tilde{\circ}\, (-(d_1 + d_2) \circ c))_\delta^{-1} \\
&= c@(d_1 + d_2).
\end{aligned}$$ ■

Next, the main result of the paper is given. Consider a SISO input-affine analytic nonlinear system (1) (so $m = \ell = 1$) having full relative degree and corresponding normal form

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= p(z) + \kappa(z)u \\
y &= z_1
\end{aligned}$$

about z_0. (Note z has been redefined in the new coordinate system.) This will be abbreviated as (p, κ, z_0). Let $\kappa_0(z_n) := \kappa(z_{1,0}, \ldots, z_{n-1,0}, z_n)$ denote a real analytic function on some open interval containing $z_{n,0}$. This function will be called *generic* in the event that $D^i\kappa_0(z_{n,0}) \neq 0$, $i \geq 0$, where D is the differential operator $\kappa_0(\partial/\partial z_n)$. This can coincide with a standard notion of genericness depending on the origin of the model. For example, if the set of Taylor series coefficients of κ_0 is a random sequence in the product space $\mathbb{R}^\omega := \Pi_{i=0}^\infty X_i$, where $X = \mathbb{R}$, one can define a product measure on $\mathbb{R}^\omega$ via any continuous probability measure on $\mathbb{R}$. In which case, the probability that $D^i\kappa_0(z_{n,0}) = 0$ for one or more i is zero. On the another hand, if κ_0 is a polynomial in z_n, then it is not generic.

Theorem 5: Suppose a SISO analytic nonlinear system (1) has relative degree $r = n$, and its corresponding normal form is (p, κ, z_0) with κ_0 being generic. Then $y = F_c[u]$, where the maximum invariant subseries of the generating series $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ under dynamic output feedback contains the subseries

$$\sum_{i=1}^\infty (c, x_0^{n-1}x_1^i)x_0^{n-1}x_1^i \tag{6}$$

with $(c, x_0^{n-1}x_1^i) \neq 0$, $i \geq 0$. Furthermore, this subseries is realized by the *reduced* normal form $(0, \kappa_0(z_n), z_0)$.

Proof: A straightforward calculation using the assumptions of full relative degree and that κ_0 is generic gives

$$(c, x_0^{n-1}x_1^i) = L_{g_1}^i L_{g_0}^{n-1}h(z_0) = D^{i-1}\kappa_0(z_{n,0}) \neq 0, \quad i \geq 1.$$

From this result it is also clear that the reduced normal form realizes the series (6). Now applying the algorithm in Theorem 2, it follows directly that c_i is equivalent to (6) in this case. Since c_i is known to be invariant under the output feedback group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, it is automatically invariant under the additive subgroup $(\mathbb{R}\langle\langle X\rangle\rangle, +, 0)$. Hence, the claim is proved. ■

It should be plainly stated that no claim is being made that subseries (6) is maximal in any sense. But to date no counterexample has been discovered to the contrary under the stated conditions, which admittedly are strong. So the question remains open.

V. Examples

Four examples are presented next. The first two examples satisfy the assumptions stated in Theorem 5. The second two examples do not and thus illustrate some consequences of this fact.

Example 2: The following system is considered in [15, p. 151]:

$$\begin{aligned}\dot{z} &= \begin{bmatrix} 0 \\ z_1 + z_2^2 \\ z_1 - z_2 \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{z_2} \\ \mathrm{e}^{z_2} \\ 0 \end{bmatrix} u \\ y &= z_3.\end{aligned}$$

The system has full relative degree $r = 3$ about any point z satisfying $1 + 2z_2 \neq 0$. At the point $z = 0$, for example, the generating series is

$$\begin{aligned}c = &-\underline{x_0^2 x_1} - \underline{3x_0^2 x_1^2} - 4x_0^3 x_1^2 - 3x_0^2 x_1 x_0 x_1 - \underline{8x_0^2 x_1^3} - \\ &4x_0^4 x_1^2 - 2x_0^3 x_1 x_0 x_1 - 24x_0^3 x_1^3 - 19x_0^2 x_1 x_0 x_1^2 - \\ &8x_0^2 x_1^2 x_0 x_1 - \underline{28x_0^2 x_1^4} - 60x_0^4 x_1^3 - 42x_0^3 x_1 x_0 x_1^2 - \\ &22x_0^3 x_1^2 x_0 x_1 - 116x_0^3 x_1^4 - 22x_0^2 x_1 x_0^2 x_1^2 - \\ &14x_0^2 x_1 x_0 x_1 x_0 x_1 - 105x_0^2 x_1 x_0 x_1^3 - 64x_0^2 x_1^2 x_0 x_1^2 - \\ &28x_0^2 x_1^3 x_0 x_1 - \underline{124x_0^2 x_1^5} - 96x_0^5 x_1^3 - 72x_0^4 x_1 x_0 x_1^2 - \\ &40x_0^4 x_1^2 x_0 x_1 - 476x_0^4 x_1^4 - 40x_0^3 x_1 x_0^2 x_1^2 - \\ &22x_0^3 x_1 x_0 x_1 x_0 x_1 - 370x_0^3 x_1 x_0 x_1^3 - 242x_0^3 x_1^2 x_0 x_1^2 - \\ &112x_0^3 x_1^3 x_0 x_1 - 620x_0^3 x_1^5 - 12x_0^2 x_1 x_0^3 x_1^2 - \\ &6x_0^2 x_1 x_0^2 x_1 x_0 x_1 - 264x_0^2 x_1 x_0^2 x_1^3 - 190x_0^2 x_1 x_0 x_1 x_0 x_1^2 - \\ &93x_0^2 x_1 x_0 x_1^2 x_0 x_1 - 620x_0^2 x_1 x_0 x_1^4 - 72x_0^2 x_1^2 x_0^2 x_1^2 - \\ &44x_0^2 x_1^2 x_0 x_1 x_0 x_1 - 448x_0^2 x_1^2 x_0 x_1^3 - 276x_0^2 x_1^3 x_0 x_1^2 - \\ &124x_0^2 x_1^4 x_0 x_1 - \underline{668x_0^2 x_1^6} - \cdots .\end{aligned}$$

A candidate for the maximum invariant series can be easily identified without any theory by simply selecting feedback systems F_d at random in Figure 1 and comparing c against $c@d$. It quickly becomes apparent from such an experiment that the terms underlined above are invariant, so that one may conjecture that the maximum invariant series under dynamic output feedback is

$$-x_0^2 x_1 - 3x_0^2 x_1^2 - 8x_0^2 x_1^3 - 28x_0^2 x_1^4 - 124x_0^2 x_1^5 - 668x_0^2 x_1^6 - \cdots .$$

The coefficients $1, 3, 8, 28, 124, 668, \ldots$ comprise the integer sequence A000776 in [19]. The normal form about $z_0 = 0$ is determined to be

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= p(z_2, z_3) + \kappa(z_2, z_3) u \\ y &= z_1,\end{aligned}$$

where

$$\begin{aligned}p(z_2, z_3) &= z_3(\sqrt{1 - 4(z_2 + z_3)} - 1) \\ \kappa(z_2, z_3) &= -\sqrt{1 - 4(z_2 + z_3)} \cdot \\ &\exp\left(\frac{1}{2}\left(\sqrt{1 - 4(z_2 + z_3)} - 1\right)\right).\end{aligned}$$

The reduced normal is therefore

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -\sqrt{1 - 4z_3} \exp\left(\frac{1}{2}\left(\sqrt{1 - 4z_3} - 1\right)\right) \\ y &= z_1.\end{aligned}$$

Is it easily checked that the integer sequence A000776 is generated by $D^i \kappa_0(0)$, $i \geq 0$, so Theorem 5 applies and its conclusion is consistent with what was observed in the experiment. $\square$

Example 3: The following system is found in [15, p. 156]:

$$\begin{aligned}\dot{z} &= \begin{bmatrix} z_3 + z_2 z_3 \\ z_1 \\ z_2 + z_1 z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 + z_2 \\ -z_3 \end{bmatrix} u \\ y &= z_1.\end{aligned}$$

The system has relative degree $r = 3$ about any point z satisfying $(1 + z_1)(1 + z_2)(1 + 2z_2) - z_1 z_3 \neq 0$. At the point $z = 0$ the generating series is

$$\begin{aligned}c = &\underline{x_0^2 x_1} + \underline{3x_0^2 x_1^2} + \underline{7x_0^2 x_1^3} + x_0^5 x_1 + 2x_0^4 x_1^2 + x_0^3 x_1 x_0 x_1 + \\ &x_0^2 x_1 x_0^2 x_1 + \underline{15x_0^2 x_1^4} + 13x_0^5 x_1^2 + 5x_0^4 x_1 x_0 x_1 + \\ &18x_0^4 x_1^3 + 3x_0^3 x_1 x_0^2 x_1 + 9x_0^3 x_1 x_0 x_1^2 + 3x_0^3 x_1^2 x_0 x_1 + \\ &3x_0^2 x_1 x_0^3 x_1 + 9x_0^2 x_1 x_0^2 x_1^2 + 3x_0^2 x_1 x_0 x_1 x_0 x_1 + \\ &3x_0^2 x_1^2 x_0^2 x_1 + \underline{31x_0^2 x_1^5} + 67x_0^5 x_1^3 + 25x_0^4 x_1 x_0 x_1^2 + \\ &5x_0^4 x_1^2 x_0 x_1 + 110x_0^4 x_1^4 + 11x_0^3 x_1 x_0^2 x_1^2 + \\ &x_0^3 x_1 x_0 x_1 x_0 x_1 + 55x_0^3 x_1 x_0 x_1^3 + 3x_0^3 x_1^2 x_0^2 x_1 + \\ &23x_0^3 x_1^2 x_0 x_1^2 + 7x_0^3 x_1^3 x_0 x_1 + 11x_0^2 x_1 x_0^3 x_1^2 + \\ &x_0^2 x_1 x_0^2 x_1 x_0 x_1 + 55x_0^2 x_1 x_0^2 x_1^3 + 3x_0^2 x_1 x_0 x_1 x_0^2 x_1 + \\ &23x_0^2 x_1 x_0 x_1 x_0 x_1^2 + 7x_0^2 x_1 x_0 x_1^2 x_0 x_1 + 7x_0^2 x_1^2 x_0^3 x_1 + \\ &23x_0^2 x_1^2 x_0^2 x_1^2 + 7x_0^2 x_1^2 x_0 x_1 x_0 x_1 + 7x_0^2 x_1^3 x_0^2 x_1 + \\ &\underline{63x_0^2 x_1^6} + \cdots .\end{aligned}$$

As in the previous example, the candidate maximum invariant series is identified empirically to be

$$x_0^2 x_1 + 3x_0^2 x_1^2 + 7x_0^2 x_1^3 + 15x_0^2 x_1^4 + 31x_0^2 x_1^5 + 63x_0^2 x_1^6 + \cdots .$$

The coefficients $1, 3, 7, 15, 31, 63, \ldots$ form the integer sequence A000225 in [19]. The corresponding normal form about $z_0 = 0$ is too complex to be displayed here, but its reduced normal form is

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4z_3} + 2z_3\right) u \\ y &= z_1.\end{aligned}$$

A direct calculation shows that the integer sequence A000225 is generated by $D^i \kappa_0(0)$, $i \geq 0$, so again Theorem 5 applies, and its conclusion is consistent with the experiment. $\square$

Example 4: Reconsider the LTI system $h(s)$ in Example 1 where $n = r = 3$ but now with nonlinear feedback F_d. The generating series for the plant F_c is

$$c = h_3 x_0^2 x_1 + h_4 x_0^3 x_1 + h_5 x_0^4 x_1 + h_6 x_0^5 x_1 + \cdots ,$$

where the first four h_k are given in (5). The normal form corresponds to (4) with $r = n$. The key observation is that $\kappa(z) = \kappa_0(z_3) = K$ is not generic, therefore Theorem 5 does *not* apply. A brute force calculation of the coefficients of $c@d$ for arbitrary d shows that $(c@d, x_0^{k-1}x_1) = h_k$, $k = 3, 4, 5, 6$, while all other coefficients are functions of d. So the maximal dynamic output feedback invariant series likely contains at least $h_3x_0^2x_1 + h_4x_0^3x_1 + h_5x_0^4x_1 + h_6x_0^5x_1$, and thus has three terms not predicted by Theorem 2. □

Example 5: This final example illustrates that the introduction of zero dynamics nullifies the conclusion of Theorem 5. Consider the following system appearing in [15, p. 167]:

$$\dot{z} = \begin{bmatrix} z_3 - z_2^3 \\ -z_2 \\ z_1^2 - z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} u$$
$$y = z_1,$$

which has relative degree $r = 2 < 3$ when $1 + 3z_2^2 \neq 0$. So Theorem 5 does not apply. In this case the generating series at $z = 0$ is

$$\begin{aligned} c = {} & \underline{x_0x_1} - \underline{x_0^2x_1} + x_0^3x_1 + \underline{6x_0x_1^3} - x_0^4x_1 - \underline{18x_0^2x_1^3} - \\ & \underline{12x_0x_1x_0x_1^2} - \underline{6x_0x_1^2x_0x_1} + x_0^5x_1 + 4x_0^4x_1^2 + \\ & 2x_0^3x_1x_0x_1 + \underline{54x_0^3x_1^3} + \underline{36x_0^2x_1x_0x_1^2} + \underline{18x_0^2x_1^2x_0x_1} + \\ & \underline{24x_0x_1x_0^2x_1^2} + \underline{12x_0x_1x_0x_1x_0x_1} + \underline{6x_0x_1^2x_0^2x_1} - x_0^6x_1 - \\ & 16x_0^5x_1^2 - 8x_0^4x_1x_0x_1 - 162x_0^4x_1^3 - 2x_0^3x_1x_0^2x_1 - \\ & 108x_0^3x_1x_0x_1^2 - 54x_0^3x_1^2x_0x_1 - 72x_0^2x_1x_0^2x_1^2 - \\ & 36x_0^2x_1x_0x_1x_0x_1 - 18x_0^2x_1^2x_0^2x_1 - 48x_0x_1x_0^3x_1^2 - \\ & 24x_0x_1x_0^2x_1x_0x_1 - 12x_0x_1x_0x_1x_0^2x_1 - \\ & 6x_0x_1^2x_0^3x_1 + \underline{96x_0^4x_1^4} + \underline{48x_0^3x_1x_0x_1^3} + \cdots . \end{aligned}$$

The maximum invariant series is determined empirically to be

$$\begin{aligned} & \underline{x_0x_1} - x_0^2x_1 + \underline{6x_0x_1^3} - 18x_0^2x_1^3 - 12x_0x_1x_0x_1^2 - \\ & 6x_0x_1^2x_0x_1 + 54x_0^3x_1^3 + 36x_0^2x_1x_0x_1^2 + 18x_0^2x_1^2x_0x_1 + \\ & 24x_0x_1x_0^2x_1^2 + 12x_0x_1x_0x_1x_0x_1 + 6x_0x_1^2x_0^2x_1 + \\ & 96x_0^4x_1^4 + 48x_0^3x_1x_0x_1^3 + \cdots , \end{aligned}$$

which is far from what is predicted by Theorem 5. So some generalization of this result needs to be investigated to address this more general case. □

VI. Conclusions and Future Work

The main objective of this paper was to explicitly describe invariants of a SISO plant described by a Chen–Fliess series under a class of dynamic output feedback laws when the plant has a generating series with both finite Lie rank and full relative degree. An additional genericness assumption regarding the normal form of the plant was also required. In particular, it was shown that the LTI case is too specialized for the main result to apply, and systems without full relative degree are beyond the scope of this analysis. Future work should explore how to generalize the main algorithm used in this paper in order to relax these strong assumptions.

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References

[1] J. Berstel and C. Reutenauer, Rational Series and Their Languages, Springer–Verlag, Berlin, 1988.
[2] R. W. Brockett, Feedback invariants for nonlinear systems, Proc. 7th IFAC World Congress, Helsinki, 1978, pp. 1115–1120.
[3] ——, Linear feedback systems and the groups of Galois and Lie, Linear Algebra Appl., 50 (1983) 45–60.
[4] R. W. Brockett and P. S. Krishnaprasad, A scaling theory for linear systems, IEEE Trans. Automat. Contr., AC-25 (1980) 197–207.
[5] A. Ferfera, Combinatoire du monoïde libre et composition de certains systèmes non linéaires, Astérisque, 75-76 (1980) 87–93.
[6] M. Fliess, Fonctionnelles causales non linéaires et indéterminées non commutatives, Bull. Soc. Math. France, 109 (1981) 3–40.
[7] W. S. Gray and L. A. Duffaut Espinosa, Feedback transformation group for nonlinear input-output systems, Proc. 52nd IEEE Conf. on Decision and Control, Florence, Italy, 2013, pp. 2570–2575.
[8] W. S. Gray, L. A. Duffaut Espinosa, and K. Ebrahimi-Fard, Faà di Bruno Hopf algebra of the output feedback group for multivariable Fliess operators, Systems Control Lett., 74 (2014) 64–73.
[9] W. S. Gray, L. A. Duffaut Espinosa, and M. Thitsa, Left inversion of analytic nonlinear SISO systems via formal power series methods, Automatica, 50 (2014) 2381–2388.
[10] W. S. Gray and K. Ebrahimi-Fard, SISO output affine feedback transformation group and its Faà di Bruno Hopf algebra, SIAM J. Control Optim., 55 (2017) 885–912.
[11] W. S. Gray, M. Thitsa, and L. A. Duffaut Espinosa, Pre-Lie algebra characterization of SISO feedback invariants, Proc. 53rd IEEE Conf. on Decision and Control, Los Angeles, CA, 2014, pp. 4807–4813.
[12] W. S. Gray and Y. Wang, Fliess operators on L_p spaces: convergence and continuity, Systems Control Lett., 46 (2002) 67–74.
[13] M. Green and D. J. N. Limebeer, Linear Robust Control, Prentice Hall, Englewood Cliffs, NJ, 1995.
[14] J. Hammer, Linear dynamic output feedback: invariants and stability, IEEE Trans. Automat. Contr., AC-28 (1983) 489–496.
[15] A. Isidori, Nonlinear Control Systems, 3rd Ed., Springer-Verlag, London, 1995.
[16] B. Jakubczyk, Equivalence and invariants of nonlinear control systems, in Nonlinear Controllability and Optimal Control, H. J. Sussmann, Ed., Marcel Dekker, New York, Basel, 1990, pp. 177–218.
[17] B. Jakubczyk and W. Respondek, On linearization of control systems, Bull. Acad. Polon. Sci. Sér. Sci. Math., 28 (1980) 517–522.
[18] R. E. Kalman, On partial realizations, transfer functions, and canonical forms, Acta Polytech. Scandinavica, 31 (1979) 9–32.
[19] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http:www.research.att.com/∼njas/sequences.
[20] I. A. Tall and W. Respondek, Feedback classification of nonlinear single-input control systems with controllable linearization: Normal forms, canonical forms, and invariants, SIAM J. Control Optim., 41 (2002) 1498–1531.