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LTI System Analysis via Conversion to Externally Positive Systems: Order Reduction via Elimination and Duplication Matrices

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***Abstract*— Recently the author has shown an analysis technique of general, not necessarily positive, LTI systems via conversion to externally positive systems. More precisely, the author established a construction method of an externally positive system whose impulse response is given by the square of the original LTI system to be analyzed. Then, it has been proved that the H_2 norm computation problem of a general LTI system of order n can be reduced into the L_∞-induced norm computation problem of an externally positive system of order n^2. On the basis of these preceding results, in this study, we show that the order of the externally positive system can be reduced up to $n(n+1)/2$ by using the elimination and duplication matrices that are intensively studied by Jan R. Magnus in the 80's. In addition to the computational complexity reduction in dealing with the H_2 analysis, we show that such construction of externally positive systems with reduced order is quite effective in semidefinite-programming-based peak value analysis of impulse responses of general LTI systems.**

Keywords: system conversion, externally positive system, order reduction, elimination/duplication matrices, peak value analysis of impulse responses.

I. Introduction

In the field of control theory, analysis and synthesis of linear time-invariant (LTI) positive systems have attracted growing attention recently. In particular, convex-optimization-based methods have been extensively studied for *internally* positive systems, where an internally positive system is characterized by the property that its state and output are nonnegative for any nonnegative initial state and nonnegative input [8], [11]. By making good use of the internal positivity property, "strong" analysis and synthesis conditions have been derived by, e.g., Rantzer [14], [15], Blanchini et al. [1], Shen and Lam [17], Valcher and Misra [20], Shorten and Mason [9], [13], [18], Tanaka and Langbort [19], Briat [3], and Ebihara et al. [6], [7]. On the other hand, an LTI system is said to be *externally* positive if its output is nonnegative for any nonnegative input under zero initial state [11], [8]. This is a milder requirement than that of the internal positivity and can be restated equivalently that the system impulse response is nonnegative. Due to this nonnegativity property, it can be easily shown that the L_∞-induced norm of an LTI SISO externally positive system can be given in a closed-form. It should be emphasized that the exact computation of L_∞-induced norm of a general (i.e., not necessarily positive) system is very hard since we need to integrate the absolute value of its impulse response. When we deal with externally positive systems, we can skip the operation to take the absolute value since the impulse response is inherently nonnegative, and this facilitates to obtain a closed-form formula for the L_∞-induced norm.

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Motivated by this nice property of externally positive systems, in [5], we studied analysis technique of general, not necessarily positive, LTI systems via conversion to externally positive systems. More precisely, we established a Kronecker-product-based construction method of an externally positive system whose impulse response is given by the square of the original LTI system to be analyzed. On the basis of this system conversion, we showed that the H_2 norm computation problem of a general LTI system of order n can be reduced into the L_∞-induced norm computation problem of an externally positive system of order n^2. By this problem reduction, we successfully derived novel closed-form formulas and semidefinite programs (SDPs) that characterize the H_2 norm of general LTI systems.

Even though the study in [5] has shown a potential ability of positive system theory to handle non-positive systems, the order n^2 of the resulting externally positive system is much higher than n of the original LTI system. This leads to the increase of computational burden in carrying out associated analysis and synthesis. With this drawback in mind, the goal of this paper is to establish a systematic method to reduce the order of the externally positive system. To this end, we actively use nice properties of the elimination and duplication matrices related to Kronecker product that are shown by Magnus [12]. By applying the projections to the coefficient matrices of the externally positive system by means of the elimination and duplication matrices, we show that the order can be reduced up to $n(n+1)/2$.

As another novel contribution over [5], in this paper, we deal with peak value analysis of the impulse responses of general LTI systems via proposed system conversion. To the best of the author's knowledge, there is no exact and tractable method for the peak value analysis, and what is available is only an SDP-based method that provides an upper bound of the peak value [2], [16]. We show that, by simply applying this SDP to the converted externally positive system with reduced order, we can obtain sharper (no looser) upper bounds.

We use the following notation. We denote by $\mathbb{R}$ and $\mathbb{R}_+$ the set of real and nonnegative real numbers, respectively. The set of Hurwitz stable matrices of size n is denoted by $\mathbb{H}^n$. The set of symmetric, positive semidefinite, and positive definite matrices of size n are denoted by $\mathbb{S}^n$, $\mathbb{S}^n_+$, and $\mathbb{S}^n_{++}$, respectively. For $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ the set of the eigenvalues of A and define $\mathrm{He}\{A\} := A + A^T$. For a vector $v \in \mathbb{R}^n$, we denote by $\|v\|_\infty$ its ∞-norm, i.e., $\|v\|_\infty =$

$\max_i |v_i|$. For a vector function $v: \mathbb{R}_+ \to \mathbb{R}^n$, we denote by $\|v\|_\infty$ its L_∞-norm, i.e., $\|v\|_\infty = \sup_{0\le t<\infty} \|v(t)\|_\infty$. For $A_1 \in \mathbb{R}^{n_1\times m_1}$ and $A_2 \in \mathbb{R}^{n_2\times m_2}$, we denote by $A_1 \otimes A_2$ their Kronecker product. For $A_1 \in \mathbb{R}^{n_1\times n_1}$ and $A_2 \in \mathbb{R}^{n_2\times n_2}$, we denote by $A_1 \oplus A_2$ their Kronecker sum, i.e., $A_1 \oplus A_2 := A_1 \otimes I_{n_2} + I_{n_1} \otimes A_2$.

II. Preliminaries and Review of [5]

A. Preliminaries

Let us consider the LTI SISO system G described by

$$G: \left\{ \begin{array}{lcl} \dot{x}(t) & = & Ax(t) + Bw(t), \\ z(t) & = & Cx(t), \end{array} \right. \tag{1}$$
$$A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times 1},\ C \in \mathbb{R}^{1\times n}.$$

The transfer function and the impulse response of the system G are given respectively by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = C(sI - A)^{-1}B, \tag{2}$$

$$g(t) = C\exp(At)B\ (t \ge 0). \tag{3}$$

The definition of external positivity for the system G and a related result are now reviewed.

Definition 1: [8], [11] The system G given by (1) is said to be externally positive if its output is nonnegative for any nonnegative input under zero initial state.

Proposition 1: [8], [11] The system G given by (1) is externally positive if and only if its impulse response g given by (3) is nonnegative, i.e., $g(t) \ge 0\ (\forall t \ge 0)$.

On the other hand, the definitions of the H_2 and the L_∞-induced norms of the system G are recalled as follows.

Definition 2: [21] Suppose the LTI system G given by (1) is asymptotically stable, i.e., $A \in \mathbb{H}^n$. Then, the H_2 norm of G is defined by

$$\|G\|_2 := \sqrt{\int_0^\infty g(t)^2 dt} \tag{4}$$

where g is the impulse response of G given by (3).

Definition 3: Suppose the LTI system G given by (1) is asymptotically stable. Then, the L_∞-induced norm of G is defined by

$$\|G\|_{\infty,\infty} := \sup_{\|w\|_\infty \le 1} \|z\|_\infty. \tag{5}$$

Here, it is elementary to verify that

$$\|G\|_{\infty,\infty} = \int_0^\infty |g(t)|dt.$$

Namely, the L_∞-induced norm $\|G\|_{\infty,\infty}$ coincides with the L_1 norm of the impulse response g. In particular, if the system G is externally positive, the above integration can be done by skipping the operation of taking the absolute value and hence we readily obtain

$$\|G\|_{\infty,\infty} = \int_0^\infty g(t)dt = -CA^{-1}B. \tag{6}$$

The relationship between (4) and (6) clearly shows that, if we can construct an externally positive and stable LTI system $G_{\rm sq}$ with impulse response g^2 from a given stable LTI system G with impulse response g, we can compute the H_2 norm $\|G\|_2$ by the closed-form formula (6) using the coefficient matrices of $G_{\rm sq}$. This is the motivation of the study in [5] and the results there are quickly reviewed in the next subsection.

B. Conversion to Externally Positive Systems: Review of [5]

The next result is the key in the study of [5].

Proposition 2: [5] Let us consider the LTI SISO systems G given by (1) with impulse response (3). Then, the LTI SISO system $G_{\rm sq}$ defined by

$$G_{\rm sq}(s) = \left[\begin{array}{c|c} A_{\rm sq} & B_{\rm sq} \\ \hline C_{\rm sq} & 0 \end{array}\right] := \left[\begin{array}{c|c} A\oplus A & B\otimes B \\ \hline C\otimes C & 0 \end{array}\right] \tag{7}$$

has the impulse response of the form

$$g_{\rm sq}(t) = g(t)^2\ (t \ge 0). \tag{8}$$

This proposition shows that we can construct an externally positive and stable LTI system $G_{\rm sq}$ with impulse response g^2 from a given stable LTI system G with impulse response g. Note that $A_{\rm sq} = A \oplus A \in \mathbb{H}^{n^2}$ holds if and only if $A \in \mathbb{H}^n$ holds. This can be readily verified since $\sigma(A_{\rm sq}) = \{\lambda_i + \lambda_j : \lambda_i, \lambda_j \in \sigma(A)\}$, see [10] for details.

By (4), (8), (6), and (7), it can be readily seen that the next result holds.

$$\begin{array}{rcl} \|G\|_2 & = & \sqrt{\|G_{\rm sq}\|_{\infty,\infty}} \\ & = & \sqrt{-(C\otimes C)(A\oplus A)^{-1}(B\otimes B)} \end{array} \tag{9}$$

Namely, we can obtain a closed-form formula for $\|G\|_2$ by means of the easily available L_∞-induced norm characterization of the externally positive system $G_{\rm sq}$. The relationship between this formula and the well-known Gramian-based characterization, and a novel SDP-based characterization of $\|G\|_2$ using the SDP-based characterization of $\|G_{\rm sq}\|_{\infty,\infty}$ are studied in detail in [5]. To summarize, the study in [5] has shown a potential ability of positive system theory to handle the H_2 analysis of non-positive systems. It is nonetheless true that the order n^2 of the externally positive system $G_{\rm sq}$ is much higher than n of the original system G. This leads to the increase of computational burden in carrying out associated analysis and synthesis. Therefore the goal of this paper is to establish a systematic method to reduce the order of the externally positive system $G_{\rm sq}$.

III. Order Reduction via Projection with Elimination and Duplication Matrices

In this section we show a systematic method to reduce the order of the system $G_{\rm sq}$ given by (7). It turns out that this can be done by applying a projection to the coefficient matrices of $G_{\rm sq}$ using the elimination and duplication matrices.

A. Elimination and Duplication Matrices

In this section, we recall the definition and basic properties of the elimination and duplication matrices by following [12].

For the definition of the elimination and duplication matrices, we need preliminaries. For $A \in \mathbb{R}^{n\times n}$, we denote by $\overline{A}$ the lower triangular matrix derived from A by setting all supradiagonal entries of A equal to zero, and ${\rm dg}(A)$ the diagonal matrix derived from A by setting all supra- and infradiagonal entries of A equal to zero. For $A = [\ a_1\ \cdots\ a_m\ ] \in \mathbb{R}^{n\times m}$ with its columns $a_i \in \mathbb{R}^n$ $(i = 1,\cdots,m)$, we denote by ${\rm vec}(A) \in \mathbb{R}^{mn}$ the column-expansion of A, i.e.,

$$\mathrm{vec}(A) := \begin{bmatrix} a_1^T & \cdots & a_m^T \end{bmatrix}^T \in \mathbb{R}^{mn}.$$

We also introduce $\mathrm{v}(A) \in \mathbb{R}^{n(n+1)/2}$ for $A \in \mathbb{R}^{n\times n}$, which is obtained from $\mathrm{vec}(A)$ by eliminating those entries a_{ij} with $i < j$. For instance, if $n = 3$, we have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3},$$
$$\mathrm{vec}(A) = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{12} & a_{22} & a_{32} & a_{13} & a_{23} & a_{33} \end{bmatrix}^T,$$
$$\mathrm{v}(A) = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{22} & a_{32} & a_{33} \end{bmatrix}^T.$$

Then, the definition of the elimination and duplication matrices can be given as follows.

Definition 4: [12] The elimination matrix $L \in \mathbb{R}^{n(n+1)/2\times n^2}$ performs for every matrix matrix $A \in \mathbb{R}^{n\times n}$ the transformation $L\mathrm{vec}(A) = \mathrm{v}(A)$. The duplication matrix $D \in \mathbb{R}^{n^2\times n(n+1)/2}$ performs for every matrix matrix $A \in \mathbb{R}^{n\times n}$ the transformation $D\mathrm{v}(A) = \mathrm{vec}(\overline{A} + \overline{A}^T - \mathrm{dg}(A))$.

For instance, if $n = 3$, we have

$$L = \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix},$$
$$D = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}.$$

We can confirm that $LD = I_{n(n+1)/2}$ and this in particular implies that L is of full-row rank and D is of full-column rank. It is also clear that for $A \in \mathbb{S}^n$ we have

$$D L \mathrm{vec}(A) = \mathrm{vec}(A). \tag{10}$$

In the next lemma we summarize the important properties of the elimination and duplication matrices in relation with Kronecker product.

Lemma 1: For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$, we have

$$DL(A \otimes A)D = (A \otimes A)D, \tag{11a}$$
$$DL(A \oplus A)D = (A \oplus A)D, \tag{11b}$$
$$DL(B \otimes B) = B \otimes B. \tag{11c}$$

In this lemma, the proofs for (11a) and (11b) are given in [12]. The validity of (11c) readily follows from (10) if we note that $B \otimes B = \mathrm{vec}(BB^T)$ for $B \in \mathbb{R}^{n\times 1}$.

B. Order Reduction of G_{sq}

The next theorem shows that we can reduce the order of G_{sq} given by (7) by applying a projection to its coefficient matrices using the elimination and duplication matrices.

Theorem 1: Let us consider the LTI SISO systems G given by (1) with impulse response (3). Then, the LTI SISO system $G_{\mathrm{sq,r}}$ defined by

$$\begin{aligned} G_{\mathrm{sq,r}}(s) &= \left[\begin{array}{c|c} A_{\mathrm{sq,r}} & B_{\mathrm{sq,r}} \\ \hline C_{\mathrm{sq,r}} & 0 \end{array}\right] \\ &:= \left[\begin{array}{c|c} L(A\oplus A)D & L(B\otimes B) \\ \hline (C\otimes C)D & 0 \end{array}\right] \\ &\left(= \left[\begin{array}{c|c} LA_{\mathrm{sq}}D & LB_{\mathrm{sq}} \\ \hline C_{\mathrm{sq}}D & 0 \end{array}\right]\right) \end{aligned} \tag{12}$$

has the impulse response of the form

$$g_{\mathrm{sq,r}}(t) = g(t)^2 (= g_{\mathrm{sq}}(t)) \ (t \geq 0). \tag{13}$$

Proof of Theorem 1: It is elementary to see that

$$\begin{aligned} \exp(A_{\mathrm{sq,r}}t) &= \exp\left(L(A\oplus A)Dt\right) \\ &= \sum_{i=0}^{\infty} \frac{(L(A\oplus A)Dt)^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{(L(A\oplus A)D)^i t^i}{i!} \\ &= L\sum_{i=0}^{\infty} \frac{(A\oplus A)^i t^i}{i!} D \\ &= L\exp((A\oplus A)t)D \\ &= L\exp(At)\otimes\exp(At)D \end{aligned}$$

where we used (11b) to verify the fourth equality. It follows that for $t \geq 0$ we have

$$\begin{aligned} g_{\mathrm{sq,r}}(t) &= C_{\mathrm{sq,r}}\exp(A_{\mathrm{sq,r}}t)B_{\mathrm{sq,r}} \\ &= (C\otimes C)DL(\exp(At)\otimes\exp(At))DL(B\otimes B) \\ &= (C\otimes C)DL(\exp(At)\otimes\exp(At))(B\otimes B) \\ &= (C\otimes C)DL((\exp(At)B)\otimes(\exp(At)B)) \\ &= (C\otimes C)((\exp(At)B)\otimes(\exp(At)B)) \\ &= (C\exp(At)B)\otimes(C\exp(At)B)) \\ &= g(t)^2 \end{aligned}$$

where we used (11c) to verify the third and the fifth equalities. This completes the proof. ∎

From this theorem, we see that the order n^2 of G_{sq} can be reduced to $n(n+1)/2$. In other words, we can construct an externally positive and stable LTI system $G_{\mathrm{sq,r}}$ of order $n(n+1)/2$ with impulse response g^2 from a given stable LTI system G of order n with impulse response g. Note that $A_{\mathrm{sq,r}} = L(A\oplus A)D \in \mathbb{H}^{n(n+1)/2}$ holds if and only if $A \in \mathbb{H}^n$ holds. This can be readily verified since $\sigma(A_{\mathrm{sq,r}}) = \{\lambda_i + \lambda_j : \lambda_i, \lambda_j \in \sigma(A), i \geq j\}$, see [12] for details.

On the basis of Theorem 1, we can derive another closed-form characterization of the H_2 norm $\|G\|_2$. Namely, by replacing G_{sq} in (9) with $G_{\mathrm{sq,r}}$, we readily obtain

$$\begin{aligned} \|G\|_2 &= \sqrt{\|G_{\mathrm{sq,r}}\|_{\infty,\infty}} \\ &= \sqrt{-(C\otimes C)D(L(A\oplus A)D)^{-1}L(B\otimes B)}. \end{aligned} \tag{14}$$

This result shows that, for the closed-form characterization of $\|G\|_2$, it is not necessarily to deal with larger size matrices $(A\oplus A, B\otimes B, C\otimes C)$ and it suffices to treat smaller size matrices $(L(A\oplus A)D, L(B\otimes B), (C\otimes C)D)$. This essentially leads to the reduction of the associated computational burden.

We finally note that in Theorem 1 we have shown that the realization $(A_{\mathrm{sq}}, B_{\mathrm{sq}}, C_{\mathrm{sq}})$ of G_{sq} given by (7) is always non-minimal. From a mathematical system theoretic point of view, it is then interesting if we can ensure the minimality of $(A_{\mathrm{sq,r}}, B_{\mathrm{sq,r}}, C_{\mathrm{sq,r}})$ of $G_{\mathrm{sq,r}}$ given by (12). Unfortunately this is not true as we see in the next subsection.

C. On the Minimality of $G_{\rm sq,r}$: A Counter Example

Even if the realization (A,B,C) of G shown in (1) is minimal, we cannot expect in general that the realization $(A_{\rm sq,r},B_{\rm sq,r},C_{\rm sq,r})$ of $G_{\rm sq,r}$ shown in (12) is minimal. This fact can readily be seen by the following counter example.

Let us consider the controllable pair (A,B) with

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix},B=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

This matrix A has eigenvalues $\lambda_1=1$, $\lambda_2=0$, and $\lambda_3=-1$. For the pair (A,B), we have from (12) that

$$A_{\rm sq,r}=\begin{bmatrix}2&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-2\end{bmatrix},B_{\rm sq,r}=\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}.$$

Then, it is clear that the pair $(A_{\rm sq,r},B_{\rm sq,r})$ is not controllable. In this example, we see that the matrix $A_{\rm sq,r}$ has eigenvalue 0 of geometric degree two, and this eigenvalue comes from the eigenvalues of A as in $\lambda_1+\lambda_3$ and $\lambda_2+\lambda_2$. From this example, we infer that the controllability under the proposed system conversion can not be preserved in such a case where a duplicated eigenvalue appears in $A_{\rm sq,r}$ from sums of the distinct eigenvalues of the original matrix A.

On the other hand, it is of course true that the proposed system conversion can generate a minimal realization. A most easy but non-trivial example would be the system G with

$$A=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\ B=\begin{bmatrix}1\\1\end{bmatrix},\ C=\begin{bmatrix}1&1\end{bmatrix}.$$

This results in the system $G_{\rm sq,r}$ with

$$A_{\rm sq,r}=\begin{bmatrix}2&0&0\\0&0&0\\0&0&-2\end{bmatrix},B_{\rm sq,r}=\begin{bmatrix}1\\1\\1\end{bmatrix},C_{\rm sq,r}=\begin{bmatrix}1&2&1\end{bmatrix}.$$

Clearly this realization is minimal.

IV. Peak Value Analysis of Impulse Responses

Suppose the LTI SISO system G given (1) is stable, i.e., $A\in\mathbb{H}^n$. In this section, we focus on its impulse response given by (3) and analyze its peak value $\|G\|_{\rm IP}$ defined by

$$\|G\|_{\rm IP}:=\max_{t\in[0,\infty)}|g(t)|. \tag{15}$$

Such peak value analysis is of practical use when dealing with those control systems where there is a saturation in control input or limitations in the magnitude of controlled signals due to some physical reasons.

To the best of the author's knowledge, there is no exact and tractable method for the computation of $\|G\|_{\rm IP}$ defined by (15). However, the next result is available for the upper bound computation of $\|G\|_{\rm IP}$.

Lemma 2: [2], [16], [4] Let us consider the LTI SISO system G described by (1) and suppose G is stable, i.e., $A\in\mathbb{H}^n$. Then, for a given $\gamma>0$, we have (i) $\Leftarrow$ (ii) with respect to the following two conditions.

(i) $\|G\|_{\rm IP}<\gamma$.

(ii) There exists $P\in\mathbb{S}^n_{++}$ such that

$$PA+A^TP\prec 0, \tag{16a}$$
$$B^TPB<\gamma^2, \tag{16b}$$
$$P-C^TC\succ 0, \tag{16c}$$

From this lemma, we see that an upper bound $\overline{\gamma}\geq\|G\|_{\rm IP}$ can be obtained by solving the following SDP:

$$\overline{\gamma}:=\inf_{\gamma>0,P\in\mathbb{S}^n_{++}}\gamma\ \text{subject to (16)}. \tag{17}$$

Since $\overline{\gamma}$ is merely an upper bound of $\|G\|_{\rm IP}$ in general, it is desirable if we can build novel SDPs in a systematic fashion so that we can narrow the gap. The next theorem shows that such SDPs are readily available by the proposed conversion to externally positive systems.

Theorem 2: Let us consider the LTI SISO system G described by (1) and suppose G is stable, i.e., $A\in\mathbb{H}^n$. Then, for a given $\gamma>0$, we have (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (i) with respect to the conditions (i) and (ii) in Lemma 2 and (iii) and (iv) given in the following.

(iii) There exists $P_{\rm sq}\in\mathbb{S}^{n^2}_{++}$ such that

$$P_{\rm sq}A_{\rm sq}+A_{\rm sq}^TP_{\rm sq}\prec 0, \tag{18a}$$
$$B_{\rm sq}^TP_{\rm sq}B_{\rm sq}<\gamma^4, \tag{18b}$$
$$P_{\rm sq}-C_{\rm sq}^TC_{\rm sq}\succ 0 \tag{18c}$$

where $(A_{\rm sq},B_{\rm sq},C_{\rm sq})$ is given by (7).

(iv) There exists $P_{\rm sq,r}\in\mathbb{S}^{n(n+1)/2}_{++}$ such that

$$P_{\rm sq,r}A_{\rm sq,r}+A_{\rm sq,r}^TP_{\rm sq,r}\prec 0, \tag{19a}$$
$$B_{\rm sq,r}^TP_{\rm sq,r}B_{\rm sq,r}<\gamma^4, \tag{19b}$$
$$P_{\rm sq,r}-C_{\rm sq,r}^TC_{\rm sq,r}\succ 0 \tag{19c}$$

where $(A_{\rm sq,r},B_{\rm sq,r},C_{\rm sq,r})$ is given by (12).

In this theorem, it should be noted that the relations (iii)$\Rightarrow$(i) and (iv)$\Rightarrow$(i) are obvious from Lemma 2 and the basic fact that $\|G\|_{\rm IP}^2=\|G_{\rm sq}\|_{\rm IP}=\|G_{\rm sq,r}\|_{\rm IP}$. The proof for the essential part (ii)$\Rightarrow$(iii)$\Leftrightarrow$(iv) is given in the appendix section.

With Theorem 2 in mind, let us define $\overline{\gamma}_{\rm sq}$ and $\overline{\gamma}_{\rm sq,r}$ by

$$\overline{\gamma}_{\rm sq}:=\inf_{\gamma>0,P_{\rm sq}\in\mathbb{S}^{n^2}_{++}}\gamma\ \text{subject to (18)}, \tag{20}$$

$$\overline{\gamma}_{\rm sq,r}:=\inf_{\gamma>0,P_{\rm sq,r}\in\mathbb{S}^{n(n+1)/2}_{++}}\gamma\ \text{subject to (19)}. \tag{21}$$

Then, it is very clear from Theorem 2 that

$$\overline{\gamma}\geq\overline{\gamma}_{\rm sq}=\overline{\gamma}_{\rm sq,r}\geq\|G\|_{\rm IP}. \tag{22}$$

Namely, we can obtain sharper (no looser) upper bounds by solving the SDPs (20) and (21) built for the externally positive systems $G_{\rm sq}$ and $G_{\rm sq,r}$, respectively. Even though these SDPs yield an exactly the same upper bound, the SDP (21) is computationally much less demanding than the SDP (20) since the size of the LMIs and the size of the variable are reduced drastically. We illustrate this efficiency via numerical examples in the next section.

V. Numerical Examples

Let us consider the case where $n=8$ and the coefficient matrices of the system G given by (1) are

$$A = \begin{bmatrix} -1.68 & 1.17 & -0.31 & 0.19 & 0.24 & -0.46 & 1.67 & 0.24 \\ 1.16 & -0.88 & 0.34 & -0.89 & -0.41 & 1.69 & -0.63 & 1.26 \\ -0.11 & 0.18 & -1.18 & -0.85 & -0.72 & -1.11 & 0.16 & 0.03 \\ 0.71 & 1.25 & -0.84 & -1.94 & -0.07 & 1.40 & 0.33 & -0.07 \\ 1.62 & -0.76 & -0.09 & -1.05 & -2.30 & -0.61 & 0.87 & -0.45 \\ -1.43 & -0.10 & 0.59 & 0.80 & -0.04 & -1.65 & 1.21 & 1.32 \\ -1.28 & -0.07 & 0.82 & 1.20 & 0.15 & 1.08 & -2.36 & -0.89 \\ -0.57 & -0.25 & 0.75 & -1.17 & -0.23 & -0.99 & 0.14 & -1.91 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.85 & -0.15 & -0.51 & -0.70 & -1.64 & -0.06 & -1.62 & 0.73 \end{bmatrix}^T,$$

$$C = \begin{bmatrix} -0.49 & -0.64 & -0.28 & -0.26 & 0.54 & 0.85 & 0.25 & -0.14 \end{bmatrix}.$$

Note that $\sigma(A) = \{-0.2023 \pm 1.5584j, -0.5121, -1.5995 \pm 1.1244j, -3.2830 \pm 0.8951j, -3.2182\}$ and hence $A \in \mathbb{H}^8$. The impulse response of the system G is shown in Fig. 1. From this figure, we see that $\|G\|_{\rm IP} \approx 0.8613$.

To confirm the validity of (22), we solved the SDPs (17), (20), (21) and obtained the upper bounds, $\overline{\gamma}$, $\overline{\gamma}_{\rm sq}$, $\overline{\gamma}_{\rm sq,r}$. In Table I we show the results together with CPU time needed to solve the corresponding SDPs. From Table I, we can confirm that (22) holds. In particular, even though the SDPs (20) and (21) yielded the same upper bound $\overline{\gamma}_{\rm sq} = \overline{\gamma}_{\rm sq,r} = 0.9054$, it is very clear that we can solve the SDP (21) much faster than (20). This clearly shows the effectiveness of the order reduction of $G_{\rm sq}$ and the construction of the reduced-order externally positive system $G_{\rm sq,r}$ by means of the elimination and duplication matrices.

VI. Conclusion

In this paper, we focused on the construction of an externally positive system whose impulse response is given by the square of that of a given LTI system of order n. On the basis of our preceding result enabling the construction of such an externally positive system of order n^2, we showed that the order can be reduced to $n(n+1)/2$ by making good use of nice properties of the elimination and duplication matrices in relation with Kronecker product. We finally illustrated that such a conversion to reduced order externally positive system is useful in computing upper bounds of the peak value of the impulse response of general, not necessarily positive, LTI SISO systems in a less conservative and efficient manner.

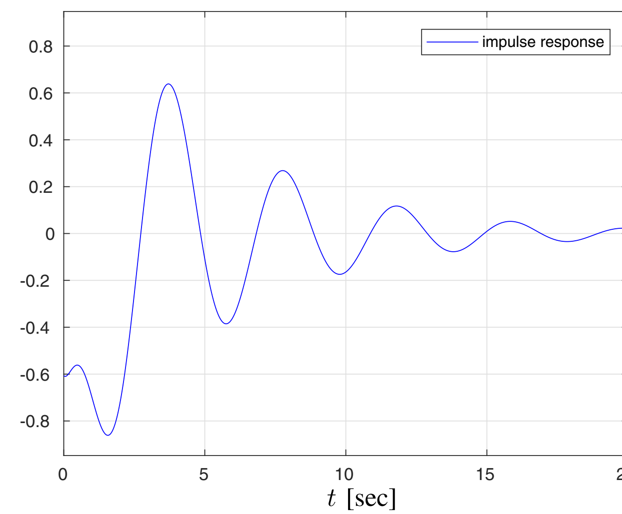

Fig. 1. Impulse Response of G.

TABLE I

Computed Upper Bounds and CPU Time.

	value	CPU time
$\overline{\gamma}$ from SDP (17)	1.2845	0.25
$\overline{\gamma}_{\rm sq}$ from SDP (20)	0.9054	28.21
$\overline{\gamma}_{\rm sq,r}$ from SDP (21)	0.9054	1.86

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Appendix I
Proof of Theorem 2

It suffices to prove (ii)⇒(iii), (iii)⇒(iv), and (iii)⇐(iv). In the following we use the very basic fact that $Q \otimes R \succ 0$

holds for $Q \succ 0$ and $R \succ 0$. We also use (11b) and (11c) frequently without clear indication.

Proof of Theorem 2:

(ii)⇒(iii) Suppose the LMI (16) is satisfied with $P = \Pi \in \mathbb{S}_{++}^n$. Then, for $\Pi_{\rm sq} := \Pi \otimes \Pi \in \mathbb{S}^{n^2}$, we have

$$\begin{aligned} \Pi_{\rm sq}A_{\rm sq} + A_{\rm sq}^T\Pi_{\rm sq} &= {\rm He}\{(\Pi\otimes\Pi)(A\otimes I + I\otimes A)\} \\ &= \Pi\otimes(\Pi A + A^T\Pi) + (\Pi A + A^T\Pi)\otimes\Pi \\ &\prec 0, \end{aligned}$$

$$\begin{aligned} B_{\rm sq}^T\Pi_{\rm sq}B_{\rm sq} &= (B\otimes B)^T(\Pi\otimes\Pi)(B\otimes B) \\ &= (B^T\Pi B)(B^T\Pi B) \\ &< \gamma^4, \end{aligned}$$

$$\begin{aligned} \Pi_{\rm sq} - C_{\rm sq}^TC_{\rm sq} &= \Pi\otimes\Pi - (C\otimes C)^T(C\otimes C) \\ &= \Pi\otimes\Pi - (C^TC)\otimes(C^TC) \\ &\succ \Pi\otimes(C^TC) - (C^TC)\otimes(C^TC) \\ &\succeq (C^TC)\otimes(C^TC) - (C^TC)\otimes(C^TC) \\ &= 0. \end{aligned}$$

It follows that (18) in (iii) is satisfied with $P_{\rm sq} = \Pi_{\rm sq}$. This completes the proof.

(iii)⇒(iv) Suppose the LMI (18) is satisfied with $P_{\rm sq} = \Pi_{\rm sq} \in \mathbb{S}_{++}^{n^2}$. Then, for $\Pi_{\rm sq,r} := D^T\Pi_{\rm sq}D \in \mathbb{S}^{n(n+1)/2}$, we have

$$\begin{aligned} \Pi_{\rm sq,r}A_{\rm sq,r} + A_{\rm sq,r}^T\Pi_{\rm sq,r} &= {\rm He}\left\{D^T\Pi_{\rm sq}DL(A\oplus A)D\right\} \\ &= {\rm He}\left\{D^T\Pi_{\rm sq}(A\oplus A)D\right\} \\ &= D^T{\rm He}\left\{\Pi_{\rm sq}A_{\rm sq}\right\}D \\ &\prec 0, \end{aligned}$$

$$\begin{aligned} B_{\rm sq,r}^T\Pi_{\rm sq,r}B_{\rm sq,r} &= (B\otimes B)^TL^TD^T\Pi_{\rm sq}DL(B\otimes B) \\ &= (B\otimes B)^T\Pi_{\rm sq}(B\otimes B) \\ &= B_{\rm sq}^T\Pi_{\rm sq}B_{\rm sq} \\ &< \gamma^4, \end{aligned}$$

$$\begin{aligned} \Pi_{\rm sq,r} - C_{\rm sq,r}^TC_{\rm sq,r} &= D^T\Pi_{\rm sq}D - D^T(C\otimes C)^T(C\otimes C)D \\ &= D^T(\Pi_{\rm sq} - (C\otimes C)^T(C\otimes C))D \\ &= D^T(\Pi_{\rm sq} - C_{\rm sq}^TC_{\rm sq}))D \\ &\succ 0. \end{aligned}$$

It follows that (19) in (iv) is satisfied with $P_{\rm sq,r} = \Pi_{\rm sq,r}$. This completes the proof.

(iii)⇐(iv) The proof for this relation can not be done straightforwardly as in the preceding two cases since essentially we have to construct a larger size matrix $P_{\rm sq} \in \mathbb{S}_{++}^{n^2}$ that satisfies the LMI (18) from a smaller size matrix $P_{\rm sq,r} \in \mathbb{S}_{++}^{n(n+1)/2}$ that satisfies the LMI (19). For the proof, suppose the LMI (19) is satisfied with $P_{\rm sq,r} = \Pi_{\rm sq,r} \in \mathbb{S}_{++}^{n(n+1)/2}$. Then, applying a congruence transformation with $\Xi_{\rm sq,r} = \Pi_{\rm sq,r}^{-1} \in \mathbb{S}_{++}^{n(n+1)/2}$, we obtain

$$ {\rm He}\{LA_{\rm sq}D\Xi_{\rm sq,r}\} \prec 0, \tag{23a}$$

$$ \begin{bmatrix} \Xi_{\rm sq,r} & LB_{\rm sq} \\ B_{\rm sq}^TL^T & \gamma^4 \end{bmatrix} \succ 0, \tag{23b}$$

$$ C_{\rm sq}D\Xi_{\rm sq,r}D^TC_{\rm sq}^T < 1. \tag{23c}$$

Here, if we define $\Xi_{\rm sq,0} := D\Xi_{\rm sq,r}D^T \in \mathbb{S}_{+}^{n^2}$, we can proceed from (23a) as

$$\begin{aligned} & D{\rm He}\{LA_{\rm sq}D\Xi_{\rm sq,r}\}D^T \preceq 0, \\ \Leftrightarrow\ & {\rm He}\{DLA_{\rm sq}D\Xi_{\rm sq,r}D^T\} \preceq 0, \\ \Leftrightarrow\ & {\rm He}\{A_{\rm sq}D\Xi_{\rm sq,r}D^T\} \preceq 0, \\ \Leftrightarrow\ & {\rm He}\{A_{\rm sq}\Xi_{\rm sq,0}\} \preceq 0. \end{aligned} \tag{24}$$

Similarly, from (23b), we obtain

$$\begin{aligned} & \begin{bmatrix} D\Xi_{\rm sq,r}D^T & DLB_{\rm sq} \\ B_{\rm sq}^TL^TD^T & \gamma^4 \end{bmatrix} \succeq 0, \\ \Leftrightarrow\ & \begin{bmatrix} \Xi_{\rm sq,0} & B_{\rm sq} \\ B_{\rm sq}^T & \gamma^4 \end{bmatrix} \succeq 0. \end{aligned} \tag{25}$$

It follows that

$$ A_{\rm sq}\Xi_{\rm sq,0} + \Xi_{\rm sq,0}A_{\rm sq}^T \preceq 0, \tag{26a}$$

$$ \begin{bmatrix} \Xi_{\rm sq,0} & B_{\rm sq} \\ B_{\rm sq}^T & \gamma^4 \end{bmatrix} \succeq 0, \tag{26b}$$

$$ C_{\rm sq}\Xi_{\rm sq,0}C_{\rm sq}^T < 1. \tag{26c}$$

Here, since $A_{\rm sq} \in \mathbb{H}^{n^2}$, there exists $\Xi_{\rm sq,\varepsilon} \in \mathbb{S}_{++}^{n^2}$ for $\varepsilon > 0$ such that

$$ A_{\rm sq}\Xi_{\rm sq,\varepsilon} + \Xi_{\rm sq,\varepsilon}A_{\rm sq}^T + \varepsilon I_{n^2} = 0.$$

Then, if we define $\Xi_{\rm sq} := \Xi_{\rm sq,0} + \Xi_{\rm sq,\varepsilon} \in \mathbb{S}_{++}^{n^2}$, there exists (sufficiently small) $\varepsilon > 0$ such that

$$ A_{\rm sq}\Xi_{\rm sq} + \Xi_{\rm sq}A_{\rm sq}^T \prec 0, \tag{27a}$$

$$ \begin{bmatrix} \Xi_{\rm sq} & B_{\rm sq} \\ B_{\rm sq}^T & \gamma^4 \end{bmatrix} \succ 0, \tag{27b}$$

$$ C_{\rm sq}\Xi_{\rm sq}C_{\rm sq}^T < 1. \tag{27c}$$

By applying a congruence transformation with $\Pi_{\rm sq} = \Xi_{\rm sq}^{-1} \in \mathbb{S}_{++}^{n^2}$, we obtain

$$ \Pi_{\rm sq}A_{\rm sq} + A_{\rm sq}^T\Pi_{\rm sq} \prec 0, \tag{28a}$$

$$ B_{\rm sq}^T\Pi_{\rm sq}B_{\rm sq} < \gamma^4, \tag{28b}$$

$$ \Pi_{\rm sq} - C_{\rm sq}^TC_{\rm sq} \succ 0. \tag{28c}$$

It follows that (18) in (iii) is satisfied with $P_{\rm sq} = \Pi_{\rm sq}$. This completes the proof. ■