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Characterizations of input-to-state practical stability for finite-dimensional and infinite-dimensional systems

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Abstract— **For a broad class of infinite-dimensional systems, we characterize input-to-state practical stability (ISpS) using the uniform limit property and in terms of input-to-state stability. We specialize our results to the systems with Lipschitz continuous flows and evolution equations in Banach spaces. Even for the special case of ordinary differential equations our characterizations of ISpS via the limit property are novel and improve existing criteria for ISpS.**

Keywords: input-to-state stability, nonlinear systems, practical stability, infinite-dimensional systems.

I. Introduction

The concept of input-to-state stability (ISS), introduced in [2], has become indispensable for various branches of nonlinear control theory, such as robust stabilization of nonlinear systems [3], design of nonlinear observers [4], analysis of large-scale networks [5], [6], etc.

However, in many cases it is impossible (as in quantized control) or too costly to construct a feedback, ensuring ISS behavior of the closed loop system. To address such applications, a relaxation of the ISS concept has been proposed in [5], called input-to-state practical stability (ISpS, practical ISS). This concept is extremely useful for stabilization of stochastic control systems [7], control under quantization errors [8], [9], study of interconnections of nonlinear systems by means of small-gain theorems [5], [10], etc.

Criteria for ISS in terms of other stability properties are among foundational theoretical results in ISS of ordinary differential equations (ODEs). In [11] Sontag and Wang have proved an ISS superposition theorem, saying that ISS is equivalent to the limit property combined with a local stability. Characterizations of ISS greatly simplify the proofs of other important results, such as small-gain theorems for ODEs [6] and hybrid systems [12], non-coercive ISS Lyapunov theorems [13], relations between ISS and nonlinear $L_2 \to L_2$ stability [14], to name a few examples.

Characterizations of ISS for ODEs in [11] heavily exploit the topological structure of an underlying state space $\mathbb{R}^n$, as well as a special type of dynamics (ODEs). Trying to generalize these criteria to infinite-dimensional systems, we face fundamental difficulties: closed bounded balls are never compact in infinite-dimensional normed linear spaces, nonuniformly globally asymptotically stable nonlinear systems do not necessarily have bounded reachability sets, and even if they do, this still does not guarantee uniform global stability [13]. These difficulties have been overcome in a recent work [13], where characterizations of ISS have been developed for a general class of control systems, encompassing evolution PDEs, differential equations in Banach spaces, time-delay systems, switched systems, ODEs, etc. The results in [13] naturally extend criteria for ISS of ODEs developed in [11]. New notions and results obtained in [13] establish a solid background for a solution of further problems. The concept of a uniform limit has been useful in the theory of non-coercive Lyapunov functions [13].

Despite a great importance of practical ISS for nonlinear control theory, much less is known about characterizations of practical ISS even in ODE setting. Sontag and Wang have shown in [11, Proposition VI.3] that an ODE system is ISpS iff it is compact ISS, i.e., there is a compact 0-invariant set $\mathscr{A} \subset \mathbb{R}^n$ so that the system has a uniform asymptotic gain (UAG) w.r.t. $\mathscr{A}$. This is an interesting characterization, but UAG itself is a quite strong property and it may be hard to check it. It would be desirable to obtain criteria for ISS in terms of weaker properties as limit property, which will be as powerful as characterizations of ISS given in [11] for ODEs and in [13] for general infinite-dimensional systems.

In this paper we develop such criteria for practical ISS for a broad class of infinite-dimensional systems. The understanding of the nature of practical ISS will be beneficial for the development of quantized and sample data controllers for infinite-dimensional systems and will give further insights into the ISS theory of infinite-dimensional systems, which is currently a hot topic [15], [16], [17], [18], [19], [13].

We prove in Section IV that *a nonlinear infinite-dimensional control system Σ possessing bounded reachability sets is practically ISS if and only if there is a bounded subset $\mathscr{A}$ of a state space so that Σ has a uniform limit property (ULIM) w.r.t. $\mathscr{A}$*. This criterion can be used to prove ISpS of control systems. On the other hand, we show that any ISpS control system has a so-called complete uniform asymptotic gain property (CUAG), which is stronger than uniform asymptotic gain property (UAG) as defined in [11], [13].

An important difference of this criterion of ISpS to the criteria of ISS proved in [11], [13] is that it does not involve any kind of stability w.r.t. the set $\mathscr{A}$ (which is necessary for ISS), which significantly simplifies verification of the ISpS property.

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The full journal version of this article is [1].

We base ourselves on machinery developed in [13] for characterization of ISS of general infinite-dimensional systems, in particular, we use the notion of the uniform limit and results from [13] related to this property. Additionally, we develop two further technical results which are of independent interest.

Firstly, *we introduce a CUAG property and show in Proposition 3.2 that a control system possesses this property if and only if it has a UAG property and if its finite time reachability sets are bounded.*

Secondly, using this CUAG characterization we show in Proposition 4.2 that *if a system has uniform limit property w.r.t. certain bounded set $\mathscr{A}$ of a state space X and if this system has bounded finite time reachability sets, then there is a set $\mathscr{B} \supset \mathscr{A}$ so that Σ has a (much stronger than ULIM) CUAG property w.r.t. $\mathscr{B}$*. In our proof we construct a family of such sets.

These results can be refined for special classes of system as systems with Lipschitz continuous flows or semilinear equations in Banach spaces. Even specialized to ODE systems our characterizations of practical ISS via the limit property are novel. In Section V we show that *an ODE system Σ is practically ISS $\Leftrightarrow$ there is a bounded set $\mathscr{A}$ so that Σ has a limit property w.r.t. $\mathscr{A}$ $\Leftrightarrow$ Σ is compact ISS*. This recovers [11, Proposition VI.3] and considerably strengthens [11, Lemma I.4].

Due to the space limitations we omit most of the proofs. They can be found in the journal version of this article [1].

A. Notation

The following notation will be used throughout these notes. Denote $\mathbb{R}_+ := [0,+\infty)$. For an arbitrary set S and $n \in \mathbb{N}$ the n-fold Cartesian product is $S^n := S \times \ldots \times S$.

Let X be a normed linear space with a norm $\|\cdot\|$ and let $\mathscr{A}$ be a nonempty set in X. For any $x \in X$ we define a distance from $x \in X$ to $\mathscr{A}$ by $\|x\|_{\mathscr{A}} := \inf_{y\in\mathscr{A}} \|x-y\|$. Define also $\|\mathscr{A}\| := \sup_{x\in\mathscr{A}} \|x\|$. The open ball in a normed linear space X with radius r around $\mathscr{A} \subset X$ is denoted by $B_r(\mathscr{A}) := \{x \in X : \ \|x\|_{\mathscr{A}} < r\}$. For short, we denote $B_r := B_r(\{0\})$. Similarly, $B_{r,\mathscr{U}} := \{u \in \mathscr{U} : \|u\|_{\mathscr{U}} < r\}$. The closure of a set $S \subset X$ w.r.t. norm $\|\cdot\|$ is denoted by $\overline{S}$. With a slight abuse of notation we define $\overline{B_0(\mathscr{A})} := \mathscr{A}$ and $\overline{B_{0,\mathscr{U}}} = \{0\}$.

For the formulation of stability properties the following classes of comparison functions are useful:

$$\begin{aligned}
\mathscr{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly increasing and } \gamma(0)=0\},\\
\mathscr{K}_\infty &:= \{\gamma \in \mathscr{K} \mid \gamma \text{ is unbounded}\},\\
\mathscr{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty}\gamma(t)=0\},\\
\mathscr{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t) \in \mathscr{K},\ \beta(r,\cdot) \in \mathscr{L},\ \forall t \ge 0,\ \forall r > 0\}.
\end{aligned}$$

II. Preliminaries

In this paper, we consider abstract axiomatically defined time-invariant and forward complete systems:

Definition 2.1: Consider the triple $\Sigma = (X, \mathscr{U}, \phi)$ consisting of

(i) A normed linear space $(X, \|\cdot\|)$, called the state space, endowed with the norm $\|\cdot\|$.
(ii) A set of input values U, which is a nonempty subset of a certain normed linear space.
(iii) A space of inputs $\mathscr{U} \subset \{f : \mathbb{R}_+ \to U\}$, $0 \in \mathscr{U}$ endowed with a norm $\|\cdot\|_{\mathscr{U}}$ satisfying two axioms:
The axiom of shift invariance states that for all $u \in \mathscr{U}$ and all $\tau \ge 0$ the time shift $u(\cdot+\tau)$ belongs to $\mathscr{U}$ with $\|u\|_{\mathscr{U}} \ge \|u(\cdot+\tau)\|_{\mathscr{U}}$.
The axiom of concatenation is defined by the requirement that for all $u_1, u_2 \in \mathscr{U}$ and for all $t > 0$ the concatenation of u_1 and u_2 at time t
$$u(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0,t],\\ u_2(\tau - t), & \text{otherwise,}\end{cases} \tag{1}$$
belongs to $\mathscr{U}$. Furthermore, if $u_2 \equiv 0$, then $\|u\|_{\mathscr{U}} \le \|u_1\|_{\mathscr{U}}$.
(iv) A transition map $\phi : \mathbb{R}_+ \times X \times \mathscr{U} \to X$.

The triple Σ is called a (forward complete) control system, if the following properties hold:

(Σ1) Forward completeness: for every $(x,u) \in X \times \mathscr{U}$ and for all $t \ge 0$ the value $\phi(t,x,u) \in X$ is well-defined.
(Σ2) The identity property: for every $(x,u) \in X \times \mathscr{U}$ it holds that $\phi(0,x,u) = x$.
(Σ3) Causality: for every $(t,x,u) \in \mathbb{R}_+ \times X \times \mathscr{U}$, for every $\tilde{u} \in \mathscr{U}$, such that $u(s) = \tilde{u}(s)$, $s \in [0,t]$ it holds that $\phi(t,x,u) = \phi(t,x,\tilde{u})$.
(Σ4) Continuity: for each $(x,u) \in X \times \mathscr{U}$ the map $t \mapsto \phi(t,x,u)$ is continuous.
(Σ5) The cocycle property: for all $t,h \ge 0$, for all $x \in X$, $u \in \mathscr{U}$ we have $\phi(h,\phi(t,x,u),u(t+\cdot)) = \phi(t+h,x,u)$.

Remark 2.2: In compare to the paper [13], upon which this note is based, we impose here an additional requirement on the space $\mathscr{U}$, that the concatenation of any input u with a zero input has the norm which is not larger than $\|u\|_{\mathscr{U}}$. This condition is satisfied by most of the "natural" input spaces.

Definition 2.3: Let a control system $\Sigma = (X, \mathscr{U}, \phi)$, a real number $s \ge 0$ and $\mathscr{A} \subset X$, $\mathscr{A} \ne \emptyset$ be given. $\mathscr{A}$ is called s-invariant if $\phi(t,x,u) \in \mathscr{A}$ for all $t \ge 0$, $x \in \mathscr{A}$ and $u \in \overline{B_{s,\mathscr{U}}}$.

The central notion of this paper is:

Definition 2.4: A control system $\Sigma = (X, \mathscr{U}, \phi)$ is called *(uniformly) input-to-state practically stable (ISpS) w.r.t. a nonempty set $\mathscr{A} \subset X$*, if there exist $\beta \in \mathscr{KL}$, $\gamma \in \mathscr{K}_\infty$ and $c > 0$ s.t. for all $x \in X$, $u \in \mathscr{U}$ and $t \ge 0$ the following holds:
$$\|\phi(t,x,u)\|_{\mathscr{A}} \le \beta(\|x\|_{\mathscr{A}}, t) + \gamma(\|u\|_{\mathscr{U}}) + c. \tag{2}$$

If ISpS property w.r.t. $\mathscr{A}$ holds with $c := 0$, then Σ is called *input-to-state stable (ISS)* w.r.t $\mathscr{A}$.

In what follows we always assume that the set, w.r.t. which the stability property is considered (usually denoted by $\mathscr{A}$) is always nonempty.

We are interested in ISpS w.r.t. bounded subsets of X:

Definition 2.5: A control system $\Sigma = (X, \mathscr{U}, \phi)$ is called *ISpS*, if there is a bounded set $\mathscr{A} \subset X$ s.t. Σ is ISpS w.r.t. $\mathscr{A}$.

Our aim is to prove criteria for practical ISS in terms of more basic stability properties, which are listed next:

Definition 2.6: A control system $\Sigma = (X, \mathcal{U}, \phi)$

- has *bounded reachability sets (BRS)*, if

$$C > 0, \ \tau > 0 \quad \Rightarrow \sup_{\|x\| \leq C, \ \|u\|_{\mathcal{U}} \leq C, \ t \in [0,\tau]} \|\phi(t,x,u)\| < \infty.$$

- has the *uniform asymptotic gain (UAG) property w.r.t.* $\mathcal{A} \subset X$, if there exists a $\gamma \in \mathcal{K}_\infty$ such that for all $\varepsilon, r > 0$ there is a $\tau = \tau(\varepsilon, r) < \infty$ s.t. for all $u \in \mathcal{U}$ and all $x \in \overline{B_r(\mathcal{A})}$

$$t \geq \tau \quad \Rightarrow \quad \|\phi(t,x,u)\|_{\mathcal{A}} \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \qquad (3)$$

- has the *limit property (LIM) w.r.t.* $\mathcal{A} \subset X$ if there is a $\gamma \in \mathcal{K}_\infty$: for all $x \in X$, $u \in \mathcal{U}$ and $\varepsilon > 0$ there is a $t = t(x,u,\varepsilon)$:

$$\|\phi(t,x,u)\|_{\mathcal{A}} \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

- has the *uniform limit property (ULIM) w.r.t.* $\mathcal{A} \subset X$, if there exists $\gamma \in \mathcal{K}_\infty$ so that for all $\varepsilon > 0$ and all $r > 0$ there is a $\tau = \tau(\varepsilon, r)$ s.t. for all $u \in \mathcal{U}$:

$$\|x\|_{\mathcal{A}} \leq r \ \Rightarrow \ \exists t \leq \tau(\varepsilon, r) : \ \|\phi(t,x,u)\|_{\mathcal{A}} \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \quad (4)$$

Remark 2.7: For ODEs forward completeness implies BRS property, see [20, Proposition 5.1]. For ∞-dimensional systems this is not always the case (see [13, Example 2]).

Note that trajectories of ULIM systems do not only approach the ball $B_{\gamma(\|u\|_{\mathcal{U}})}(\mathcal{A})$ (as trajectories of LIM systems do), but they do it uniformly. Indeed, the time of approachability τ depends only on the norm of the state and ε and does not depend on he state itself.

III. Complete uniform asymptotic gains

Uniform asymptotic gain property assures that the trajectories possess a uniform convergence rate. However, UAG property per se does not guarantee that the solutions possess any kind of uniform global bounds (one can construct examples of control systems, illustrating this fact, using ideas from [13, Example 2]). Since it is often desirable both to have uniform attraction rates as well as uniform bounds on solutions, we introduce (motivated by [21, Definition 4.1.3], where a similar concept with $\gamma = 0$ has been employed) a new notion:

Definition 3.1: We say that a control system Σ satisfies the completely uniform asymptotic gain property (CUAG) w.r.t. $\mathcal{A} \subset X$, if there are $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and $C > 0$ s.t. for all $x \in X$, $u \in \mathcal{U}$, $t \geq 0$ it holds that:

$$\|\phi(t,x,u)\|_{\mathcal{A}} \leq \beta(\|x\|_{\mathcal{A}} + C, t) + \gamma(\|u\|_{\mathcal{U}}). \qquad (5)$$

The next proposition gives a useful criterion for CUAG.

Proposition 3.2: Let $\mathcal{A} \subset X$ be a bounded set. A control system Σ is CUAG w.r.t. $\mathcal{A} \Leftrightarrow \Sigma$ is BRS and UAG w.r.t. $\mathcal{A}$.

For our purposes CUAG is important due to the following

Proposition 3.3: If there is a bounded $\mathcal{A} \subset X$ so that a control system Σ is CUAG w.r.t. $\mathcal{A}$, then Σ is ISpS.

IV. Characterizations of ISpS

The main result of this contribution is the following characterization of ISpS:

Theorem 4.1: Let Σ be a control system as in Definition 2.1. The following statements are equivalent:

(i) Σ is ISpS

(ii) There is a bounded 0-invariant set $\mathcal{A} \subset X$ so that Σ is CUAG w.r.t. $\mathcal{A}$.

(iii) Σ is BRS and there is a bounded set $\mathcal{A} \subset X$ so that Σ is ULIM w.r.t. $\mathcal{A}$.

Theorem 4.1 can be used in two ways. On the one hand, to prove ISpS of a system, we can merely check the conditions in item (iii) of Theorem 4.1. On the other hand, if a certain system is ISpS, item (ii) shows that it enjoys also a CUAG property w.r.t. a certain bounded 0-invariant set.

The proof of this fact relies on three results: characterizations of ISS for infinite-dimensional systems achieved in [13], the results on the novel CUAG property in Section III and on the novel technique of which can be called "improving stability properties via enlarging of the attracting set", which we briefly describe next.

Assume that a control system Σ is ULIM w.r.t a certain set $\mathcal{A} \subset X$. Then it is clear that Σ is ULIM w.r.t. any $\mathcal{B} \supset \mathcal{A}$. However, it may exist certain subsets of X w.r.t. which Σ has better properties than merely ULIM. In this section we show that this is indeed the case provided Σ is BRS.

Assume that $\Sigma = (X, \mathcal{U}, \phi)$ is given. For any $\mathcal{A} \subset X$, any $\varepsilon > 0$ and any $\gamma \in \mathcal{K}_\infty$ define

$$\mathcal{A}_{\varepsilon,\gamma} := \{\phi(t,x,u) : t \in \mathbb{R}_+, x \in B_\varepsilon(\mathcal{A}), \|u\|_{\mathcal{U}} \leq \gamma^{-1}(\tfrac{\varepsilon}{2})\}. \quad (6)$$

Note that from the identity axiom (Σ2), for each $\varepsilon > 0$ and any $\gamma \in \mathcal{K}_\infty$ it holds that $\mathcal{A} \subset B_\varepsilon(\mathcal{A}) \subset \mathcal{A}_{\varepsilon,\gamma}$. The construction of the sets $\mathcal{A}_{\varepsilon,\gamma}$ is motivated by the notion of the positive prolongation of a set, see [22].

The next proposition is the central technical result needed to show Theorem 4.1.

Proposition 4.2: Assume that Σ is a BRS control system and Σ has the ULIM property w.r.t. a bounded (not necessarily 0-invariant) set $\mathcal{A} \subset X$, with $\gamma \in \mathcal{K}_\infty$ as in (4). Then for any $\varepsilon > 0$ the set $\mathcal{A}_{\varepsilon,\gamma}$ is bounded, 0-invariant and Σ is CUAG w.r.t. $\mathcal{A}_{\varepsilon,\gamma}$.

Proof: The proof goes step by step: first we show boundedness of $\mathcal{A}_{\varepsilon,\gamma}$, then 0-invariance of $\mathcal{A}_{\varepsilon,\gamma}$ and finally we show that Σ is UAG w.r.t. $\mathcal{A}_{\varepsilon,\gamma}$. Since Σ is BRS, by Proposition 3.2 it follows that Σ is CUAG w.r.t. $\mathcal{A}_{\varepsilon,\gamma}$. ■

Finally, we can prove the main result of this paper:

Proof: **(of Theorem 4.1)**

(i) $\Rightarrow$ (ii). Assume, that Σ is an ISpS control system. Then there are $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and $c > 0$ so that for all $x \in X$, $t \geq 0$ and $u \in \mathcal{U}$ we have

$$\|\phi(t,x,u)\| \leq \beta(\|x\|, t) + \gamma(\|u\|_{\mathcal{U}}) + c. \qquad (7)$$

For any $y \in X$ and any $c > 0$ it holds that $\|y\|_{\overline{B_c(0)}} = \max\{\|y\| - c, 0\}$ and $\|y\| \leq \|y\|_{\overline{B_c(0)}} + c$ and we infer from (7) for $\phi(t,x,u) \notin \overline{B_c(0)}$ that

$$\|\phi(t,x,u)\|_{\overline{B_c(0)}} \leq \beta(\|x\|_{\overline{B_c(0)}} + c, t) + \gamma(\|u\|_{\mathcal{U}}). \qquad (8)$$

Otherwise, if $\phi(t,x,u) \in \overline{B_c(0)}$, then $\|\phi(t,x,u)\|_{\overline{B_c(0)}} = 0$ and (8) also holds. Thus Σ is CUAG w.r.t. $\overline{B_c(0)}$ (however, $\overline{B_c(0)}$ does not have to be 0-invariant). According to Proposition 4.2 there is a bounded 0-invariant set $\mathcal{A}$ so that Σ is CUAG w.r.t. $\mathcal{A}$.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Proposition 4.2 implies that there is a bounded set $\mathscr{A} \subset X$ so that Σ is CUAG w.r.t. $\mathscr{A}$. Proposition 3.3 shows ISpS of Σ. ■

V. ISpS of ODE systems

The criteria for ISpS can be strengthened for some particular classes of systems as systems with Lipshcitz continuous flows, semilinear equations in Banach spaces etc. Due to the space limitations we do not discuss these results here and refer to the full journal version of this article [1]. Instead we would like to stress our attention on the results for systems of ODEs. Even specialized to this class of systems our results are stronger than other ones existing in the literature.

Consider the system given by

$$\dot{x} = f(x,u), \tag{9}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous w.r.t. the first argument and inputs u belong to the set $\mathscr{U} := L_\infty(\mathbb{R}_+,\mathbb{R}^m)$ of Lebesgue measurable globally essentially bounded functions with values in $\mathbb{R}^m$. For this class of systems the characterizations of ISpS developed in the previous sections can be considerably strengthened.

The following result has been shown in [13] for $\mathscr{A} = \{0\}$ on the basis of [11, Corollary III.3]. The proof for general bounded $\mathscr{A}$ is analogous.

Proposition 5.1: Consider a system (9) with $\mathscr{U}$ as above. Let $\mathscr{A} \subset \mathbb{R}^n$ be any bounded set. Then Σ is ULIM w.r.t. $\mathscr{A}$ if and only if Σ is LIM w.r.t. $\mathscr{A}$.

Sontag and Wang defined in [11, Section VI] the following concept:

Definition 5.2: (9) is called compact ISS, if there is a compact 0-invariant set $\mathscr{A} \subset \mathbb{R}^n$ s.t. (9) is UAG w.r.t. $\mathscr{A}$.

Criteria for practical ISS of a system (9) take a particularly simple form:

Corollary 5.3: Let (9) be forward complete. The following statements are equivalent:

(i) (9) is ISpS
(ii) For any $s > 0$ there is a compact s-invariant set $\mathscr{A} \subset \mathbb{R}^n$: (9) is ISS w.r.t. $\mathscr{A}$.
(iii) (9) is compact-ISS.
(iv) There is a bounded set $\mathscr{A} \subset \mathbb{R}^n$: (9) is LIM w.r.t. $\mathscr{A}$.

Proof: The proof is based on several results which are specific for ODE systems:

- For ODEs LIM and ULIM properties coincide, see [13]
- Forward complete systems (9) are also BRS, see [20, Proposition 5.1].
- The fact that 0-invariant spaces constructed in Proposition 4.2 are s-invariant for a suitable $s > 0$. This in turn helps to show a kind of robustness of these sets.

We omit the details and again refer to [1]. ■

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