23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

Input-to-State Stability with Respect to Different Boundary Disturbances for Burgers' Equation

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Abstract— **This paper presents two methods for the establishment of ISS estimates in L^q-norm ($q \geq 2$) for Burgers' equation with different types of boundary disturbances. Precisely, for Burgers' equation with Dirichlet boundary conditions, we use De Giorgi iteration and Lyapunov method by adequately splitting the original problem into two subsystems to establish the ISS estimates in L^q-norm with some $q \geq 2$. Whereas, for Burgers' equation with certain nonlinear boundary conditions involving spacial derivatives of the solution, we obtain the ISS estimates in L^2-norm and L^q-norm with any $q \geq 2$ by some variations of Sobolev embedding inequalities that can be used to deal with the boundary items involved in Lyapunov functionals-based analysis.**

I. Introduction

In the past few years, there has been a considerable effort devoted to extending the input-to-state stability (ISS) theory to infinite dimensional systems governed by, e.g., partial differential equations (PDEs). In particular, significant progresses on the establishment of ISS properties with respect to disturbances distributed over the domain and acting on the boundaries for different PDEs have been reported in the recent literature. The notable accomplishments include the establishment of different ISS characteristics for infinite-dimensional systems [3], [4], [5], [8], [19], [20], [22], the application of spectral decomposition and finite-difference for the *a priori* estimates of ISS [9], [10], [11], [12], [13], the monotonicity-based method for studying the ISS of nonlinear parabolic equations with boundary disturbances [21], the construction of ISS-Lyapunov functionals for certain classes of PDEs [1], [2], [16], [17], [18], [25], [26], [28], [29].

It is interesting to note that the extension of the notion of ISS to infinite dimensional systems w.r.t. in-domain disturbances is somehow straightforward, except for some technical issues and the particularity of infinite dimensional systems, e.g., general arguments may not be easily established for generic settings. However, the investigation on the ISS properties w.r.t. boundary disturbances is much more challenging [10], [11]. In particular, the well-developed Lyapunov theory may be readily applied to dealing with in-domain disturbances [17], [25]. While the successful application of this technique to the establishment of ISS properties w.r.t. boundary disturbances expressed in its original form, i.e. without involving the time-derivatives of the disturbances, has been reported in the literature only very recently [26], [28], [29]. These results show that the method of Lyapunov functionals is still effective for establishing the ISS properties for some linear and nonlinear PDEs with different types of boundary disturbances.

The aim of this paper is to illustrate the application of two methods developed in [28] and [29] in dealing with ISS estimates for Burgers' equation with Dirichlet, or Neumann or Robin, or certain nonlinear boundary conditions. For Burgers' equation with Dirichlet boundary conditions, the ISS estimate in L^2-norm has already been obtained in [28] by the method of Lyapunov functionals and De Giorgi iteration. In the present work, the ISS estimates are extended to L^q-norm for some $q \geq 2$ by using the same method. For the case where the system is subject to nonlinear Robin (or Neumann) boundary conditions, the technique based on the method of Lyapunov functionals and some Sobolev embedding-like inequalities developed in [29] for dealing with the ISS properties in L^2-norm for a class of semi-linear parabolic PDEs with Neumann or Robin boundary conditions has been applied, which leads to the ISS estimates in L^q-norm for any $q \geq 2$. Note that the present work focuses on dealing with boundary disturbances. Whereas, in-domain disturbances can be incorporated into the ISS estimates in very similar ways as shown in [28] and [29].

The rest of the paper is organized as follows. Section II introduces briefly the technique of De Giorgi iteration and presents some Sobolev embedding-like inequalities needed for the subsequent development. Section III presents the considered problems and the main results. Detailed development on the establishment of ISS properties for Burgers' equation with Dirichlet and some nonlinear Robin (or Neumann) boundary conditions are given, respectively, in Section IV and Section V. Finally, some concluding remarks are provided in Section VI.

II. Preliminaries

A. De Giorgi iteration

De Giorgi iteration, also known as De Giorgi-Nash-Moser theorem, is an important tool for regularity analysis of elliptic and parabolic PDEs [7], [23], [24]. Specifically, let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be an open bounded set and γ be a constant. The De Giorgi class $DG^+(\Omega, \gamma)$ consists of functions $u \in W^{1,2}(\Omega)$ which satisfy, for every ball $B_r(y) \subset \Omega$, every $0 < r' < r$, and every $k \in \mathbb{R}$, the following Caccioppoli type inequality:

$$\int_{B_{r'}(y)} |\nabla(u-k)_+|^2 \mathrm{d}x \leq \frac{\gamma}{(r-r')^2} \int_{B_r(y)} |(u-k)_+|^2 \mathrm{d}x,$$

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where $(u-k)_+ = \max\{u-k, 0\}$. The class $DG^-(\Omega,\gamma)$ is defined in a similar way. The main idea of De Giorgi iteration is to estimate $|A_k|$, the measure of $\{x \in \Omega; u(x) \geq k\}$, and derive $|A_k| = 0$ with some k for functions u in De Giorgi class.

To apply the method of De Giorgi iteration, we need to split the original system with Dirichlet boundary conditions into two subsystems, one of which is with the boundary disturbance and a zero initial value. The solution of this subsystem belongs to some De Giorgi class satisfying Caccioppoli type inequality with $\gamma = 0$ and k associated with the boundary disturbance. A consequence of that is $|A_k| = 0$, which yields the boundedness of the solution of this subsystem.

B. Preliminary inequalities

Throughout this paper, for notational simplicity, we always denote $\|\cdot\|_{L^2(0,1)}$ by $\|\cdot\|$. Let $\mathbb{R}_{\geq 0} = [0,+\infty), \mathbb{R}_+ = (0,+\infty)$.

We present below some variations of Sobolev embedding inequalities needed for the subsequent development.

Lemma 1: [29] Suppose that $u \in C^1([a,b];\mathbb{R})$. The following inequalities hold:

(i) $u^2(c) \leq \frac{2}{b-a}\|u\|^2 + (b-a)\|u_x\|^2$ for any $c \in [a,b]$;

(ii) $\|u\|^2 \leq 2u^2(c)(b-a) + (b-a)^2\|u_x\|^2$ for any $c \in [a,b]$;

(iii) $\|u\|^2 \leq \frac{(b-a)^2}{2}\|u_x\|^2$ provided $u(c_0) = 0$ for some $c_0 \in [a,b]$.

Remark 1: The Sobolev embedding-like inequalities (i) and (ii) (or (iii)) are developed in [29] for dealing with the items associated with boundary points. These inequalities are essential for the establishment of ISS properties w.r.t. boundary disturbances of linear and nonlinear systems with Robin or Neumann boundary conditions, or some nonlinear types boundary conditions involving spacial derivatives of the trajectories.

In the subsequent development, we employ extensively the following inequalities.

Young's inequality Let $1 < p, q < +\infty, \frac{1}{p} + \frac{1}{q} = 1$. There holds

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \ \forall a, b \in \mathbb{R}_{\geq 0}, \forall \varepsilon > 0,$$

where $C_\varepsilon = q^{-1}(\varepsilon p)^{-\frac{q}{p}}$.

Gronwall's inequality[6, Appendix B.2.j] Suppose that $y : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is absolutely continuous on $[0,T]$ for any $T > 0$ and satisfies for a.e. $t \geq 0$ the following differential inequality

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) \leq g(t)y(t) + h(t),$$

where $g, h \in L^1([0,T];\mathbb{R})$ for any $T > 0$. Then for all $t \in \mathbb{R}_{\geq 0}$,

$$y(t) \leq y(0)e^{\int_0^t g(s)\mathrm{d}s} + \int_0^t h(s)e^{\int_s^t g(s)\mathrm{d}s}\mathrm{d}s.$$

III. Problem Formulation and Main Results

A. Problem formulation

In this paper, we address the ISS properties for Burgers' equation

$$u_t - \alpha u_{xx} + \beta u u_x = 0, \quad \text{in } (0,1) \times \mathbb{R}_{\geq 0}, \tag{1}$$

with the initial data

$$u(x,0) = u^0(x).$$

We consider two types of boundary conditions:
(i) Dirichlet boundary conditions

$$u(0,t) = 0, u(1,t) = d_1(t), \tag{2}$$

and
(ii) nonlinear Robin (or Neumann) boundary conditions

$$u_x(0,t) = k_0\left((m_0+1)u(0,t) + u^3(0,t)\right) + d_0(t), \tag{3a}$$
$$u_x(1,t) = k_1\left((m_1+1)u(1,t) + u^3(1,t)\right) + d_1(t), \tag{3b}$$

where $d_0(t)$ and $d_1(t)$ are disturbances on the boundaries. In general, they can represent actuation and sensing errors.

Throughout this paper, we always assume that $\alpha > 0, \beta > 0, k_0, k_1, m_0, m_1$ are constants, and $d_0, d_1 \in C^2(\mathbb{R}_{\geq 0})$. Furthermore, assume that $d_1(0) = 0$ in (2).

Remark 2: Under appropriate assumptions on the initial data u^0 (e.g., $u^0 \in C^{2+\sigma}([0,1])$), the existence of the unique solution $u \in H^{2+\sigma,1+\frac{\sigma}{2}}([0,1]\times\mathbb{R}_{\geq 0})(\sigma \in (0,1))$ to (1) and (2) is guaranteed by Theorem 6.1 of [14, P. 452], where $H^{l,\frac{l}{2}}([0,1]\times\mathbb{R}_{\geq 0})$ is some Banach space of functions $u(x,t)$ introduced in [14, P. 8]. For the existence of the unique classical solution to (1), (3a) and (3b), one may proceed in a similar way as in Section 3 of [15] to consider the transformation $\widehat{u} = u - a(x)d_1(t) - b(x)d_0(t)$, where $a(x)$ and $b(x)$ are sufficiently smooth functions satisfying $a(0,t) = a(1,t) = a_x(0,t) = b(0,t) = b(1,t) = b_x(1,t) = 0, a_x(1,t) = b_x(0,t) = 1$. We have then

$$\begin{aligned}
&\widehat{u}_t - \alpha\widehat{u}_{xx} + \beta(\widehat{u} + a(x)d_1(t) + b(x)d_0(t)) \\
&\qquad\qquad \times (\widehat{u}_x + a_x(x)d_1(t) + b_x(x)d_0(t)) = 0, \\
&\widehat{u}(x,0) = u^0(x) - a(x)d_1(0) - b(x)d_0(0), \\
&\widehat{u}_x(0,t) = k_0\left((m_0+1)\widehat{u}(0,t) + \widehat{u}^3(0,t)\right), \\
&\widehat{u}_x(1,t) = k_1\left((m_1+1)\widehat{u}(1,t) + \widehat{u}^3(1,t)\right).
\end{aligned}$$

The existence of the unique solution $u \in H^{2+\sigma,1+\frac{\sigma}{2}}([0,1]\times\mathbb{R}_{\geq 0})(\beta \in (0,1))$ to (1) and (3) is guaranteed by Theorem 7.4 of [14, P. 491].

B. Main results

Let $\mathcal{K} = \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \gamma(0) = 0, \gamma$ is continuous, strictly increasing$\}$; $\mathcal{K}_\infty = \{\theta \in \mathcal{K} |\ \lim_{s\to\infty}\theta(s) = \infty\}$; $\mathcal{L} = \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \gamma$ is continuous, strictly decreasing, $\lim_{s\to\infty}\gamma(s) = 0\}$; $\mathcal{KL} = \{\mu : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \mu(\cdot,t) \in \mathcal{K}, \forall t \in \mathbb{R}_{\geq 0}$, and $\mu(s,\cdot) \in \mathcal{L}, \forall s \in \mathbb{R}_+\}$.

Let u^0 be the initial state of (1) in a certain function space $\mathcal{H}$ with norm $\|\cdot\|_{\mathcal{H}}$.

Definition 1: System (1) is said to be ISS in L^q-norm ($q \geq 1$) w.r.t. the boundary disturbance $d_i(t)$ ($i = 0, 1$), if there exist functions $\mu \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}$ ($i = 0, 1$) such that for any $t \geq 0$, the solution of (1) and (2) (or (3)) satisfies ($i = 0, 1$)

$$\|u\|_{L^q(0,1)} \leq \mu\left(\|u^0\|_{L^q(0,1)}, t\right) + \gamma_i\left(\max_{s\in[0,t]} |d_i(s)|\right). \quad (4)$$

Moreover, System (1) is said to be EISS w.r.t. the boundary disturbances $d_i(t)$, $i = 0, 1$, if there exist $\mu' \in \mathcal{K}_\infty$ and a constat $\lambda > 0$ such that $\mu(\|u^0\|_{L^q(0,1)}, t) \leq \mu'(\|u^0\|_{L^q(0,1)})e^{-\lambda t}$ in (4).

The ISS properties w.r.t. boundary disturbances for Systems (1) are stated in the following theorems.

Theorem 2: Suppose that that $\max_{t\in\mathbb{R}_{\geq 0}} |d_1(t)| \leq \frac{1}{p}\frac{\alpha}{\beta}$ for some $p \geq 1$. System (1) with (2) is EISS in L^{2p}-norm w.r.t. the boundary disturbance $d_1(t)$ having the following estimate for any $t > 0$:

$$\begin{aligned}&\|u(\cdot,t)\|^{2p}_{L^{2p}(0,1)}\\ &\quad\leq 2^{2p-1}\|u^0\|^{2p}_{L^{2p}(0,1)}e^{-\lambda t} + 2^{2p-1}\max_{s\in[0,t]}|d_1(s)|^{2p},\end{aligned}$$

where $\lambda = 2(2-\sqrt{2})\frac{2p-1}{p}\alpha > 0$.

Theorem 3: Suppose that the constants α, β, k_0, k_1, m_0, and m_1 satisfy

$$\begin{cases}\alpha = \alpha_0 + \alpha_1, \alpha_0 > 2\alpha_1, \alpha_1 > 0,\\ k_1 \leq -\dfrac{\beta}{3\alpha} < 0, m_1 > \dfrac{\alpha_1}{\alpha k_1},\\ k_0 \geq \dfrac{\beta}{3\alpha} > 0, m_0 > \dfrac{2\alpha_0}{\alpha k_0}.\end{cases}$$

(i) System (1) with (3) is EISS in L^2-norm w.r.t. the boundary disturbances $d_1(t)$ and $d_0(t)$ having the following estimate for any $t > 0$:

$$\begin{aligned}\|u(\cdot,t)\|^2 \leq& \|u^0\|^2 e^{-2(\alpha_0-2\alpha_1)t} + \frac{\alpha}{2\varepsilon(\alpha_0-2\alpha_1)}\\ &\times\left(\max_{s\in[0,t]}|d_1(s)|^2 + \max_{s\in[0,t]}|d_0(s)|^2\right),\end{aligned}$$

where ε is a positive constant satisfying (20).

(ii) Assume further that $m_1 \geq 0$, then for any $p \geq 1$, the system (1) with (3) is EISS in L^{2p}-norm w.r.t. the boundary disturbances $d_1(t)$ and $d_0(t)$ having the following estimate for any $t > 0$:

$$\begin{aligned}\|u(\cdot,t)\|^{2p}_{L^{2p}(0,1)} \leq& \|u^0\|^{2p}_{L^{2p}(0,1)}e^{-\frac{2(\alpha_0-2\alpha_1)(2p-1)}{p}t}\\ &+\frac{\alpha p^2 C_\varepsilon}{(\alpha_0-2\alpha_1)(2p-1)}\\ &\times\left(\max_{s\in[0,t]}|d_1(s)|^{2p} + \max_{s\in[0,t]}|d_0(s)|^{2p}\right),\end{aligned}$$

where $\varepsilon > 0$ satisfies (25), and $C_\varepsilon = \frac{1}{2p}\left(\frac{2p-1}{2p\varepsilon}\right)^{\frac{2p-1}{2p}}$.

Taking the $2p$th root of both sides of the inequality in (ii) of Theorem 3 and letting $p \to +\infty$, one may obtain L^∞-estimate of the solution.

Corollary 4: Under the same assumptions as in Theorem 3, the solution of the system (1) with (3) is bounded with the following estimate for any $t > 0$:

$$\|u(\cdot,t)\|_{L^\infty(0,1)} \leq \|u^0\|_{L^\infty(0,1)} + \max_{s\in[0,t]}|d_1(s)| + \max_{s\in[0,t]}|d_0(s)|.$$

Remark 3: The ISS estimate provided in Theorem 2 is an extension of the one in L^2-norm given in [29]. Nevertheless, the ISS estimate in L^∞-norm is not established yet in Theorem 2 and Theorem 3.

Remark 4: In general, the boundedness of the disturbances in Theorem 2 is a reasonable assumption for nonlinear PDEs in the establishment of ISS properties [18]. Moreover, as the aim of the work is to present two methods in the establishment of ISS properties for nonlinear equations, some assumptions on the parameters in Theorem 2 and Theorem 3 are mostly of technical nature. Nevertheless, the ISS estimates may still hold under some relaxed conditions.

IV. ISS PROPERTIES FOR BURGERS' EQUATION WITH DIRICHLET BOUNDARY CONDTIONS

In this section, we establish the ISS properties for Burgers' equation w.r.t. the boundary disturbance $d_1(t)$ described in (2) and Theorem 2.

Let w be the unique solution of the following system:

$$w_t - \alpha w_{xx} + \beta w w_x = 0 \text{ in } (0,1)\times\mathbb{R}_{\geq 0}, \quad (5a)$$
$$w(0,t) = 0, w(1,t) = d_1(t), \quad (5b)$$
$$w(x,0) = 0. \quad (5c)$$

Let $v = u - w$. Then, it is easy to see that v is the solution of the following system:

$$v_t - \alpha v_{xx} + \beta v v_x + \beta(wv)_x = 0 \text{ in } (0,1)\times\mathbb{R}_{\geq 0},$$
$$v(0,t) = v(1,t) = 0, \quad (6a)$$
$$v(x,0) = u^0(x). \quad (6b)$$

For system (5), we have the following estimates.

Theorem 5: For every $t > 0$, there holds

$$\max_{(x,s)\in[0,1]\times[0,t]}|w(x,s)| \leq \max_{s\in[0,t]}|d_1(s)|. \quad (7)$$

It follows that for any $p \geq 1$ and every $t > 0$,

$$\|w(\cdot,t)\|^{2p}_{L^{2p}(0,1)} \leq \max_{s\in[0,t]}|d_1(s)|^{2p}.$$

For system (6), we have the following estimates.

Theorem 6: Assume that $\max_{t\in\mathbb{R}_{\geq 0}}|d_1(t)| \leq \frac{1}{p}\frac{\alpha}{\beta}$ for some $p \geq 1$. For every $t > 0$, there holds

$$\|v(\cdot,t)\|^{2p}_{L^{2p}(0,1)} \leq \|u^0\|^{2p}_{L^{2p}(0,1)}e^{-\lambda t},$$

where $\lambda = 2\left(2-\sqrt{2}\right)\frac{2p-1}{p}\alpha > 0$.

Proof: [Proof of Theorem 5] We resort to the technique of De Giorgi iteration by following the standard process presented in, e.g., [27, Theorem 4.2.1] and [28].

For any fixed $t > 0$, let $k = \max\left\{\max_{s\in[0,t]}d_1(s), 0\right\} \geq 0$. Let $\eta(x,s) = (w(x,s)-k)_+\chi_{[t_1,t_2]}(s)$, where $\chi_{[t_1,t_2]}(t)$ is the character function on $[t_1, t_2]$ and $0 \le t_1 < t_2 \le t$. Multiplying (5) by η, we get

$$\int_0^t \int_0^1 (w-k)_t (w-k)_+ \chi_{[t_1,t_2]}(s) \mathrm{d}x\mathrm{d}s + \alpha \int_0^t \int_0^1 |((w-k)_+)_x|^2 \chi_{[t_1,t_2]}(s) \mathrm{d}x\mathrm{d}s + \beta \int_0^t \int_0^1 w w_x (w-k)_+ \chi_{[t_1,t_2]}(s) \mathrm{d}x\mathrm{d}s = 0. \quad (8)$$

Let $I_k(s) = \int_0^1 (w(x,s)-k)_+^2 \mathrm{d}x$, which is absolutely continuous on $[0,t]$. Suppose that $I_k(t_0) = \max_{s\in[0,t]} I_k(s)$ with some $t_0 \in [0,t]$. Due to $I_k(0) = 0$ and $I_k(s) \ge 0$, one may assume that $t_0 > 0$ without loss of generality.

For $\varepsilon > 0$ small enough, choosing $t_1 = t_0 - \varepsilon$ and $t_2 = t_0$, it follows

$$\frac{1}{2\varepsilon}\int_{t_0-\varepsilon}^{t_0} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (\widetilde{v}-k)_+^2 \mathrm{d}x\mathrm{d}s + \frac{\alpha}{\varepsilon}\int_{t_0-\varepsilon}^{t_0}\int_0^1 |((\widetilde{v}-k)_+)_x|^2 \mathrm{d}x\mathrm{d}s + \frac{\beta}{\varepsilon}\int_{t_0-\varepsilon}^{t_0}\int_0^1 w w_x (w-k)_+ \mathrm{d}x\mathrm{d}s \le 0.$$

Note that

$$\frac{1}{2\varepsilon}\int_{t_0-\varepsilon}^{t_0} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (\widetilde{v}-k)_+^2 \mathrm{d}x\mathrm{d}s = \frac{1}{2\varepsilon}(I_k(t_0) - I_k(t_0-\varepsilon)) \ge 0.$$

We have

$$\frac{\alpha}{\varepsilon}\int_{t_0-\varepsilon}^{t_0}\int_0^1 |((w-k)_+)_x|^2 \mathrm{d}x\mathrm{d}s + \frac{\beta}{\varepsilon}\int_{t_0-\varepsilon}^{t_0}\int_0^1 w w_x (w-k)_+ \mathrm{d}x\mathrm{d}s \le 0.$$

Letting $\varepsilon \to 0^+$, we get

$$\alpha \int_0^1 |((w(x,t_0)-k)_+)_x|^2 \mathrm{d}x + \beta \int_0^1 w(x,t_0) w_x(x,t_0)(w(x,t_0)-k)_+ \mathrm{d}x \le 0. \quad (9)$$

Now we write $w = w(x,t_0)$ for simplicity. Due to $(w(0,t_0)-k)_+ = (w(1,t_0)-k)_+ = 0$, one may get

$$\begin{aligned}&\int_0^1 w w_x (w-k)_+ \mathrm{d}x \\ =&\int_0^1 (w-k)_+ ((w-k)_+)_x (w-k)_+ \mathrm{d}x + \int_0^1 k((w-k)_+)_x (w-k)_+ \mathrm{d}x \\ =&\frac{1}{3}((w-k)_+)^3|_{x=0}^{x=1} + \frac{k}{2}((w-k)_+)^2|_{x=0}^{x=1} = 0. \quad (10)\end{aligned}$$

Then for $I_k(t_0)$, we get by (iii) of Lemma 1, (9) and (10)

$$I_k(t_0) = \alpha \int_0^1 |(w(x,t_0)-k)_+)|^2 \mathrm{d}x \le \frac{\alpha}{2}\int_0^1 |(w(x,t_0)-k)_+)_x|^2 \mathrm{d}x \le 0.$$

Recalling the definition of $I_k(t_0)$, for any $s \in [0,t]$ we conclude that

$$I_k(s) \le I_k(t_0) \le 0, \quad (11)$$

which implies that for almost every $(x,s) \in [0,1]\times[0,t]$ there holds

$$w(x,s) \le \max\Big\{\max_{s\in[0,t]} d_1(s), 0\Big\}.$$

By continuity of $w(x,s)$, for every $(x,s) \in [0,1]\times[0,t]$ there holds

$$w(x,s) \le \max\{\max_{s\in[0,t]} d_1(s), 0\}. \quad (12)$$

We should prove the lower boundedness of $w(x,t)$. Indeed, setting $\widetilde{w} = -w$, we get

$$\begin{aligned}&\widetilde{w}_t - \alpha \widetilde{w}_{xx} - \beta \widetilde{w}\widetilde{w}_x = 0,\\ &\widetilde{w}(0,t) = 0, \widetilde{w}(1,t) = -d_1(t),\\ &\widetilde{w}(x,0) = 0.\end{aligned}$$

Then for every $(x,s) \in [0,1]\times[0,t]$ there holds

$$-w(x,s) = \widetilde{w}(x,s) \le \max\Big\{\max_{s\in[0,t]} -d_1(s), 0\Big\}. \quad (13)$$

Finally, we conclude (7) by (12) and (13). ■

Proof: [Proof of Theorem 6] Multiplying (6) by v^{2p-1} and integrating over $(0,1)$, we get

$$\int_0^1 v_t v^{2p-1} \mathrm{d}x + \alpha(2p-1)\int_0^1 (v^{p-1}v_x)^2 \mathrm{d}x + \beta(2p-1)\int_0^1 v^{2(p-1)} v_x v w \mathrm{d}x = 0.$$

By Young's inequality, Theorem 5, and the assumption on d_1, we deduce that

$$\begin{aligned}&\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\|v^p\|^2 + \alpha(2p-1)\|v^{p-1}v_x\|^2 \\ =&-\beta(2p-1)\int_0^1 v^{2(p-1)} v_x v w \mathrm{d}x \\ \le&\frac{\beta(2p-1)}{2\varepsilon}\max_{(x,s)\in[0,1]\times[0,t]} |w(x,s)| \int_0^1 v^{2(p-1)} v_x^2 \mathrm{d}x + \frac{\varepsilon\beta(2p-1)}{2}\max_{(x,s)\in[0,1]\times[0,t]} |w(x,s)| \int_0^1 v^{2(p-1)} v^2 \mathrm{d}x \\ \le&\frac{\beta(2p-1)}{2\varepsilon}\max_{s\in[0,t]} |d_1(s)| \|v^{p-1}v_x\|^2 + \frac{\varepsilon\beta(2p-1)}{2}\max_{s\in[0,t]} |d_1(s)| \|v^p\|^2,\end{aligned}$$

which yields

$$\frac{1}{2p(2p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|v^p\|^2 \le \Big(\frac{\beta}{2\varepsilon}\max_{s\in[0,t]}|d_1(s)| - \alpha\Big)\|v^{p-1}v_x\|^2 + \frac{\varepsilon\beta}{2}\max_{s\in[0,t]}|d_1(s)|\|v^p\|^2.$$

Note that by (iii) of Lemma 1, one has

$$\|v^p\|^2 \leq \frac{p^2}{2}\|v^{p-1}v_x\|^2. \tag{14}$$

It follows

$$\frac{1}{2p(2p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|v^p\|^2 \\ \leq\left(\frac{\beta}{2}\left(\frac{1}{\varepsilon}+\frac{\varepsilon p^2}{2}\right)\max_{s\in[0,t]}|d_1(s)|-\alpha\right)\|v^{p-1}v_x\|^2. \tag{15}$$

Choosing $\varepsilon = \frac{\sqrt{2}}{p}$ and recalling the assumption on $d_1(t)$, one has

$$\frac{\beta}{2}\left(\frac{1}{\varepsilon}+\frac{\varepsilon p^2}{2}\right)\max_{s\in[0,t]}|d_1(s)| = \frac{p\beta}{\sqrt{2}}\max_{s\in[0,t]}|d_1(s)| \\ \leq\frac{p\beta}{\sqrt{2}}\times\frac{\alpha}{p\beta}=\frac{\sqrt{2}\alpha}{2}. \tag{16}$$

Then by (14), (15) and (16), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^{2p}_{L^{2p}(0,1)} =& \frac{\mathrm{d}}{\mathrm{d}t}\|v^p\|^2\\ &\leq 2p(2p-1)\left(\frac{\sqrt{2}}{2}-1\right)\alpha\|v^{p-1}v_x\|^2\\ &\leq -2p(2p-1)\left(1-\frac{\sqrt{2}}{2}\right)\alpha\times\frac{2}{p^2}\|v^p\|^2\\ &\leq -\lambda\|v^p\|^2,\end{aligned}$$

where $\lambda = 2(2-\sqrt{2})\frac{2p-1}{p}\alpha > 0$.

One may conclude the desired result by Gronwall' inequality. ■

Proof: [Proof of Theorem 2] Note that $u = w+v$. Then, we get by Theorem 5 and Theorem 6 that for $q \geq 1$:

$$\begin{aligned}&\|u(\cdot,t)\|^{2q}_{L^{2q}(0,1)}\\ \leq&2^{2q-1}\|w(\cdot,t)\|^{2q}_{L^{2q}(0,1)}+2^{2q-1}\|v(\cdot,t)\|^{2q}_{L^{2q}(0,1)}\\ \leq&2^{2q-1}\|u^0\|^2e^{-\lambda' t}+2^{2q-1}\left(\max_{s\in[0,t]}|d_1(s)|\right)^{2q},\end{aligned}$$

where $\lambda' = 2(2-\sqrt{2})\frac{2p-1}{p}\alpha > 0$. ■

Remark 5: Considering the original system (1) with (2) and applying De Giorgi iteration, one may proceed as in the proof of Theorem 5 to obtain the boundedness of the solution having the estimate

$$\max_{(x,s)\in[0,1]\times[0,t]}|u(x,s)| \leq \max_{x\in[0,1]}|u^0(x)| + \max_{s\in[0,t]}|d_1(s)|.$$

Indeed, it suffices to set $k = \max\left\{\max_{s\in[0,t]} d_1(s), u^0(x), 0\right\} \geq 0$ and take $\eta(x,s) = (u(x,s)-k)_+\chi_{[t_1,t_2]}(s)$ as a test function to obtain the boundedness from above.

Remark 6: De Giorgi iteration can be used in problems with multidimensional spacial variables, e.g.,

$$\begin{aligned}&u_t-\alpha\Delta u+\beta(uu_x+uu_y)=0, \quad \text{in} \quad \Omega\times\mathbb{R}_{\geq 0},\\ &u(x,t)=d(t), \quad \text{in} \quad \partial\Omega\times\mathbb{R}_{\geq 0},\\ &u(x,0)=u_0(x), \quad \text{in} \quad \Omega,\end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with smooth boundary $\partial\Omega$, $c \geq 0$ is a constant, and Δ is the Laplace operator.

Remark 7: De Giorgi iteration can also be used in the establishment of ISS properties for Burgers' equation with a distributed disturbance $f(x,t)$ under the form (see [28]):

$$\begin{aligned}&u_t-\alpha u_{xx}+\beta uu_x=f(x,t), \quad \text{in} \quad (0,1)\times\mathbb{R}_{\geq 0},\\ &u(0,t)=0, u(1,t)=d_1(t),\\ &u(x,0)=u_0(x).\end{aligned}$$

In such a problem, an iteration formula (see Lemma 1 in [28]) will be needed to perform the De Giorgi iteration.

V. ISS Properties for Burgers' Equation with nonlinear boundary conditions

In this section, we establish the ISS properties by Lyapunov method for Burgers' equation w.r.t. the boundary disturbances $d_0(t)$ and $d_1(t)$ described in (3) and Theorem 3. Inequalities (i) and (ii) (or (iii)) of Lemma 1 are essential for establishing the ISS properties w.r.t. boundary disturbances in such problems.

Proof: [Proof of Theorem 3] For simplicity, we write $u_i = u(i,t)$ for $i = 0, 1$.

First, we prove the ISS estimate in L^2-norm, i.e., the claim (i). Multiplying (1) with u and integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 u_t u\mathrm{d}x-\alpha\int_0^1 u_{xx}u\mathrm{d}x+\beta\int_0^1 u^2u_x\mathrm{d}x\\ =&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2-\alpha k_1(u_1^2+u_1^4)+\frac{\beta}{3}u_1^3-\alpha k_1m_1u_1^2\\ &+\alpha k_0(u_0^2+u_0^4)-\frac{\beta}{3}u_0^3+\alpha k_0m_0u_0^2\\ &-\alpha d_1(t)u_1+\alpha d_0(t)u_0+\alpha\|u_x\|^2\\ =&0.\end{aligned}$$

Note that

$$\begin{aligned}&\alpha=\alpha_0+\alpha_1, \alpha k_1\leq-\frac{\beta}{3},\ -\alpha k_0\leq-\frac{\beta}{3},\\ &u_0^3\leq u_0^2+u_0^4,\ -u_1^3\leq u_1^2+u_1^4.\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+(\alpha_0+\alpha_1)\|u_x\|^2\\ =&-\frac{\beta}{3}u_1^3+\alpha k_1(u_1^2+u_1^4)+\frac{\beta}{3}u_0^3-\alpha k_0(u_0^2+u_0^4)\\ &+\alpha d_1(t)u_1-\alpha d_0(t)u_0+\alpha k_1m_1u_1^2-\alpha k_0m_0u_0^2\\ \leq&-\frac{\beta}{3}u_1^3-\frac{\beta}{3}(u_1^2+u_1^4)+\frac{\beta}{3}u_0^3-\frac{\beta}{3}(u_0^2+u_0^4)\\ &+\alpha d_1(t)u_1-\alpha d_0(t)u_0+\alpha k_1m_1u_1^2-\alpha k_0m_0u_0^2\\ \leq&\alpha d_1(t)u_1-\alpha d_0(t)u_0+\alpha k_1m_1u_1^2-\alpha k_0m_0u_0^2\\ \leq&\frac{\alpha}{2\varepsilon}d_1^2(t)+\frac{\varepsilon}{2}u_1^2+\frac{\alpha}{2\varepsilon}d_0^2(t)+\frac{\varepsilon}{2}u_0^2\\ &+\alpha k_1m_1u_1^2-\alpha k_0m_0u_0^2\\ =&\left(\frac{\varepsilon}{2}+\alpha k_1m_1\right)u_1^2+\left(\frac{\varepsilon}{2}-\alpha k_0m_0\right)u_0^2\end{aligned}$$

$$+\frac{\alpha}{2\varepsilon}d_1^2(t)+\frac{\alpha}{2\varepsilon}d_0^2(t), \tag{17}$$

where in the last inequality we used Young's inequality.

By (i) and (ii) of Lemma 1, we have

$$\begin{aligned}\|u_x\|^2 &\geq u_1^2-2\|u\|^2,\\ \|u_x\|^2 &\geq \|u\|^2-2u_0^2.\end{aligned} \tag{18}$$

Then we infer from $\alpha_0>0,\alpha_1>0$, (17) and (18) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\alpha_1u_1^2-2\alpha_0u_0^2+(\alpha_0-2\alpha_1)\|u\|^2\\ \leq&\frac{\alpha}{2\varepsilon}d_1^2(t)+\frac{\alpha}{2\varepsilon}d_0^2(t)+\left(\frac{\varepsilon}{2}+\alpha k_1m_1\right)u_1^2\\ &+\left(\frac{\varepsilon}{2}-\alpha k_0m_0\right)u_0^2.\end{aligned} \tag{19}$$

Note that by the assumptions, we have

$$\alpha_0-2\alpha_1>0,\alpha k_1m_1<\alpha_1,-\alpha k_0m_0<-2\alpha_0.$$

One may choose $\varepsilon>0$ small enough such that

$$\frac{\varepsilon}{2}+\alpha k_1m_1\leq\alpha_1,\frac{\varepsilon}{2}-\alpha k_0m_0\leq-2\alpha_0. \tag{20}$$

Then we deduce from (19) and (20) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2\leq-2(\alpha_0-2\alpha_1)\|u\|^2+\frac{\alpha}{\varepsilon}d_1^2(t)+\frac{\alpha}{\varepsilon}d_0^2(t). \tag{21}$$

By (21) and Gronwall's inequality, one may obtain

$$\begin{aligned}\|u\|^2\leq&\|u^0\|^2e^{-2(\alpha_0-2\alpha_1)t}\\ &+\frac{\alpha}{\varepsilon}\int_0^t d_1^2(s)e^{-2(\alpha_0-2\alpha_1)t+2(\alpha_0-2\alpha_1)s}\mathrm{d}s\\ &+\frac{\alpha}{\varepsilon}\int_0^t d_0^2(s)e^{-2(\alpha_0-2\alpha_1)t+2(\alpha_0-2\alpha_1)s}\mathrm{d}s\\ \leq&\|u^0\|^2e^{-2(\alpha_0-2\alpha_1)t}\\ &+\frac{\alpha}{2\varepsilon(\alpha_0-2\alpha_1)}\left(\max_{s\in[0,t]}d_1^2(s)+\max_{s\in[0,t]}d_0^2(s)\right).\end{aligned}$$

Now we prove the ISS estimate in L^{2p}-norm for any $p\geq1$, i.e., the claim (ii). Multiplying (1) with u^{2p-1} and integrating over $[0,1]$, we have

$$\begin{aligned}&\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\|u^p\|^2-\alpha u_xu^{2p-1}|_{x=0}^{x=1}+\alpha(2p-1)\int_0^1u^{2(p-1)}u_x^2\mathrm{d}x\\ &+\frac{\beta}{2p+1}u^{2p+1}|_{x=0}^{x=1}=0.\end{aligned}$$

Considering the boundary conditions, one may get

$$\begin{aligned}&\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\|u^p\|^2+\alpha(2p-1)\|u^{p-1}u_x\|^2\\ =&u_1^{2p}\left(\alpha k_1(1+u_1^2)-\frac{\beta}{2p+1}u_1\right)\\ &+u_0^{2p}\left(-\alpha k_0(1+u_0^2)+\frac{\beta}{2p+1}u_0\right)\\ &+\alpha k_1m_1u_1^{2p}-\alpha k_0m_1u_0^{2p}+\alpha d_1u_1^{2p-1}-\alpha d_0u_0^{2p-1}\\ \leq&\underbrace{u_1^{2p}\left(\alpha k_1+\frac{\beta}{2p+1}\right)(1+u_1^2)}_{=I_1}\\ &+\underbrace{u_0^{2p}\left(-\alpha k_0+\frac{\beta}{2p+1}\right)(1+u_0^2)}_{=I_2}\\ &+\alpha k_1m_1u_1^{2p}-\alpha k_0m_1u_0^{2p}+\alpha d_1u_1^{2p-1}-\alpha d_0u_0^{2p-1}.\end{aligned}$$

Note that by assumptions, for any $p\geq1$, we have

$$\begin{aligned}\alpha k_1+\frac{\beta}{2p+1}\leq\alpha k_1+\frac{\beta}{3}\leq0,\\ -\alpha k_0+\frac{\beta}{2p+1}\leq-\alpha k_0+\frac{\beta}{3}\leq0.\end{aligned}$$

It follows $I_1\leq0$ and $I_2\leq0$. Thus

$$\begin{aligned}&\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\|u^p\|^2+\alpha(2p-1)\|u^{p-1}u_x\|^2\\ \leq&\alpha k_1m_1u_1^{2p}-\alpha k_0m_1u_0^{2p}+\alpha d_1u_1^{2p-1}-\alpha d_0u_0^{2p-1}\\ \leq&\alpha k_1m_1u_1^{2p}-\alpha k_0m_1u_0^{2p}\\ &+\varepsilon\alpha\left(u_1^{2p}+u_0^{2p}\right)+C_\varepsilon\alpha\left(d_1^{2p}+d_0^{2p}\right)\\ =&\alpha\left(k_1m_1+\varepsilon\right)u_1^{2p}+\alpha\left(-k_0m_0+\varepsilon\right)u_0^{2p}\\ &+C_\varepsilon\alpha\left(d_1^{2p}+d_0^{2p}\right),\end{aligned} \tag{22}$$

where in the second inequality we used Young's inequality with $\varepsilon>0,C_\varepsilon=\frac{1}{2p}\left(\frac{2p-1}{2p\varepsilon}\right)^{\frac{2p-1}{2p}}$.

By (i) and (ii) of Lemma 1, we have

$$\begin{aligned}u_1^{2p}-2\|u^p\|^2\leq\|(u^p)_x\|^2=p^2\|u^{p-1}u_x\|^2,\\ \|u^p\|^2-2u_0^{2p}\leq\|(u^p)_x\|^2=p^2\|u^{p-1}u_x\|^2.\end{aligned} \tag{23}$$

Then we infer from $\alpha=\alpha_0+\alpha_1$, (22) and (23) that

$$\begin{aligned}&\frac{1}{2p}\frac{\mathrm{d}}{\mathrm{d}t}\|u^p\|^2+\frac{\alpha_1(2p-1)}{p^2}u_1^{2p}-\frac{2\alpha_0(2p-1)}{p^2}u_0^{2p}\\ \leq&\alpha\left(k_1m_1+\varepsilon\right)u_1^{2p}+\alpha\left(-k_0m_0+\varepsilon\right)u_0^{2p}\\ &+C_\varepsilon\alpha\left(d_1^{2p}+d_0^{2p}\right)-\frac{(\alpha_0-2\alpha_1)(2p-1)}{p^2}\|u^p\|^2.\end{aligned} \tag{24}$$

Note that by the assumptions, we have

$$\begin{aligned}&\alpha_0-2\alpha_1>0,\\ &\alpha k_1m_1\leq0<\frac{\alpha_1(2p-1)}{p^2},\\ &-\alpha k_0m_0<-2\alpha_0\leq-\frac{2\alpha_0(2p-1)}{p^2}.\end{aligned}$$

One may choose $\varepsilon>0$ small enough such that

$$\alpha(\varepsilon+k_1m_1)\leq\frac{\alpha_1(2p-1)}{p^2}, \tag{25a}$$

$$\alpha(\varepsilon-k_0m_0)\leq-\frac{2\alpha_0(2p-1)}{p^2}. \tag{25b}$$

Then we deduce from (24) and (25)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^{2p}_{L^{2p}(0,1)}=\frac{\mathrm{d}}{\mathrm{d}t}\|u^p\|^2\\ \leq&2p\alpha C_\varepsilon\left(d_1^{2p}+d_0^{2p}\right)-\frac{2\left(\alpha_0-2\alpha_1\right)(2p-1)}{p}\|u^p\|^2\end{aligned}$$

$$=2p\alpha C_\varepsilon\left(d_1^{2p}+d_0^{2p}\right)-\frac{2(\alpha_0-2\alpha_1)(2p-1)}{p}\|u\|_{L^{2p}(0,1)}^{2p}. \tag{26}$$

By (26) and Gronwall's inequality, one may obtain the desired result. ∎

Remark 8: Under appropriate assumptions on the parameters, Lemma 1 can be used to establish ISS estimates for Burgers' equation with boundary conditions having the following types:

$$\text{(i)}\begin{cases}u(0,t)=0,\\ u_x(1,t)=k_1\Big((m_1+1)u(1,t)+u^3(1,t)\Big)+d_1(t),\end{cases}$$

$$\text{(ii)}\begin{cases}\Big(u_x(0,t)-k_0u(0,t)-\dfrac{\beta}{3\alpha}u^2(0,t)\Big)u(0,t)=0,\\ u_x(1,t)=k_1u(1,t)+\dfrac{\beta}{3\alpha}u^2(1,t)+d_1(t),\end{cases}$$

$$\text{(iii)}\begin{cases}u_x(0,t)=&k_0\Big((m_0+1)u(0,t)+u^3(0,t)\Big)\\ &+k_2(u(0,t)+u^p(0,t))+d_0(t),\\ u_x(1,t)=&k_1\Big((m_1+1)u(1,t)+u^3(1,t)\Big)\\ &+k_3(u(1,t)+u^q(1,t))+d_1(t),\end{cases}$$

where $k_0>0,k_1<0,k_2\geq 0,k_3\leq 0$ are constants, $p,q\geq 5$ are odd numbers.

We consider ISS property in L^2-norm for Burgers' equation (1) with the following boundary conditions as an example (one of the cases in (ii)).

$$\begin{cases}u(0,t)=0,\\ u_x(1,t)=k_1u(1,t)+\dfrac{\beta}{3\alpha}u^2(1,t)+d_1(t).\end{cases}$$

In this case, we assume that

$$k_1<-2,$$

then (1) is EISS.

Indeed, proceeding as before, one may get

$$\int_0^1 u_tu\mathrm{d}x-\alpha\int_0^1 u_{xx}u\mathrm{d}x+\beta\int_0^1 u^2u_x\mathrm{d}x=0,$$

which and the boundary conditions give

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\alpha\|u_x\|^2&=\alpha k_1u_1^2+\alpha d_1(t)u_1\\ &\leq\alpha k_1u_1^2+\frac{\alpha}{2\varepsilon}d_1^2(t)+\frac{\varepsilon}{2}u_1^2,\forall\varepsilon>0.\end{aligned}$$

By (ii) of Lemma 1, we have

$$\|u_x\|^2\geq\|u\|^2-2u_1^2.$$

Then one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2+\alpha\|u\|^2-2\alpha u_1^2\leq\left(\alpha k_1+\frac{\varepsilon}{2}\right)u_1^2+\frac{\alpha}{2\varepsilon}d_1^2(t).$$

Choosing $\varepsilon>0$ small enough such that

$$\alpha k_1+\frac{\varepsilon}{2}\leq-2\alpha,$$

one may obtain the EISS estimates of (1) in L^2-norm by Gronwall's inequality.

VI. Concluding Remarks

This paper considered the establishment of ISS properties for Burgers' equation with different types of boundary disturbances, including Dirichlet boundary conditions and some nonlinear boundary conditions involving spacial derivatives of the solutions. The two methods used in this work are all based on Lyapunov functionals, while combining with, respectively, the De Giogi iteration and some Sobolev embedding-like inequalities. The results of this work show that the well-established method of Lyapunov functionals remains a convenient tool for the study of the ISS property of PDEs w.r.t. boundary disturbances, as in the case where the disturbances are distributed over the domain, although its application in the former case may be very challenging and remains an open research problem.

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