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SMALL-GAIN STABILITY ANALYSIS OF PARABOLIC-HYPERBOLIC PDE LOOPS

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1. Introduction

Parabolic-hyperbolic PDE loops present unique features because they combine the finite signal transmission speed of hyperbolic PDEs with the unlimited signal transmission speed of parabolic PDEs. Since there are many possible interconnections that can be considered, it is difficult to give results for a "general case". The present work focuses on two particular cases.

The first case is the feedback interconnection of a parabolic PDE with a special first-order hyperbolic PDE: a zero-speed hyperbolic PDE. However, the study of this particular loop is of interest because it arises in the movement of chemicals underground. Moreover, the study of this system can be used for the analysis of wave equations with Kelvin-Voigt damping.

The second case is the feedback interconnection of a parabolic PDE with a first-order hyperbolic PDE by means of a combination of boundary and in-domain terms. The interconnection is effected by linear, non-local terms. The study of this case is motivated by the boundary feedback stabilization problem for a system of hyperbolic, first-order PDEs. The study of robustness of the closed-loop system with respect to neglected diffusion phenomena leads to this specific parabolic-hyperbolic PDE loop. The obtained results for this case can also be used for the derivation of delay-independent stability conditions for parabolic PDEs with delayed terms.

For both cases, we provide results for existence/uniqueness of solutions as well as sufficient conditions for ISS or exponential stability in the spatial sup norm. The proof procedures of the obtained results are similar in both cases: the small-gain methodology is used in both cases for the derivation of stability results. However, due to space limitations all proofs are omitted.

2. Movement of Chemicals Underground

Certain chemicals are released at position $\xi = 0$ and enter the groundwater system. Let $\phi \in (0,1)$ be the porosity of the soil, $v \geq 0$ be the velocity of the bulk movement of the groundwater, $c(t,\xi)$ and $n(t,\xi)$ be the concentration of the chemicals dissolved in water and the sorbed concentration of chemicals in the soil, respectively, at position $\xi \in [0,L]$ (horizontal coordinate) and time $t \geq 0$. Taking into account diffusion, the mass balance for the chemical gives the equation (see [3]):

$$\frac{\partial c}{\partial t}(t,\xi) = \frac{D}{\phi}\frac{\partial^2 c}{\partial \xi^2}(t,\xi) - v\frac{\partial c}{\partial \xi}(t,\xi) - \frac{1}{\phi}\frac{\partial n}{\partial t}(t,\xi), \text{ for } (t,\xi) \in (0,+\infty)\times(0,L) \tag{1}$$

We assume that the concentration of chemicals dissolved in underground water at $\xi = 0$ is time-varying and takes values around a nominal value $c_0 > 0$. At $\xi = L$ the ground meets the sea, where the concentration of chemicals is zero. Therefore, we obtain the boundary conditions:

$$c(t,0) - c_0 - \tilde{d}(t) = c(t,L) = 0 \text{, for all } t \geq 0 \tag{2}$$

where $\tilde{d} : \Re_+ \to [-c_0, +\infty)$ is the variation of the concentration of chemicals dissolved in water at the source ($\xi = 0$). The model of the process also includes the empirical relation (see [3]):

$$\frac{\partial n}{\partial t}(t,\xi) = a\,c(t,\xi) - bn(t,\xi) \text{, for } (t,\xi) \in (0,+\infty) \times (0,L) \tag{3}$$

where $a, b > 0$ are constants. Substituting (3) into (1) and defining

$$u_1(t,z) := c_0^{-1} \exp\left(-\frac{vL\phi}{2D} z\right)\left(c\left(\frac{L^2\phi}{D} t, Lz\right) - c_{eq}(Lz)\right),\ u_2(t,z) := c_0^{-1} \exp\left(-\frac{vL\phi}{2D} z\right)\left(n\left(\frac{L^2\phi}{D} t, Lz\right) - n_{eq}(Lz)\right),$$

we obtain from (1), (2) and (3) for $(t,z) \in (0,+\infty) \times (0,1)$:

$$\frac{\partial u_1}{\partial t}(t,z) - \frac{\partial^2 u_1}{\partial z^2}(t,z) + K\,u_1(t,z) - r\tilde{b}\,u_2(t,z) = \frac{\partial u_2}{\partial t}(t,z) - \tilde{a}\,u_1(t,z) + \tilde{b}\,u_2(t,z) = 0 \tag{4}$$

$$u_1(t,0) - d(t) = u_1(t,1) = 0 \tag{5}$$

where $d(t) := c_0^{-1}\tilde{d}\left(L^2\phi t / D\right)$, $\tilde{a} := aL^2\phi / D$, $\tilde{b} := bL^2\phi / D$, $r := \phi^{-1}$, $K := L^2\left(v^2\phi^2 + 4aD\right)/\left(4D^2\right)$. System (4), (5) is the feedback interconnection of a parabolic PDE with a first-order zero-speed hyperbolic PDE (or an infinitely-parameterized scalar ODE). Its dynamical behavior is very different from that of a parabolic PDE: notice that system (4), (5) may be transformed to a wave equation with Kelvin-Voigt damping that may also include viscous damping and stiffness terms (see [2])

$$\frac{\partial^2 u_1}{\partial t^2}(t,z) + (\tilde{b} + K)\frac{\partial u_1}{\partial t}(t,z) = \frac{\partial^3 u_1}{\partial z^2 \partial t}(t,z) + \tilde{b}\frac{\partial^2 u_1}{\partial z^2}(t,z) + \tilde{b}\left(r\tilde{a} - K\right)u_1(t,z) \tag{6}$$

and, any wave equation with Kelvin-Voigt damping can be transformed to the form (4), (5).

We next provide existence/uniqueness results for the initial-boundary value (4), (5) with

$$u_1[0] = u_{1,0},\ \ u_2[0] = u_{2,0} \tag{7}$$

where $u_{1,0}, u_{2,0}$ are real functions on $[0,1]$. Our main result is the following theorem.

Theorem 2.1: *Consider the initial-boundary value problem (4), (5), (7), where* $K, r, \tilde{a}, \tilde{b} \in \Re$ *are constants. For every* $u_{2,0} \in C^1([0,1])$, $u_{1,0} \in \left\{ w \in H^3(0,1) : w(1) = w''(0) = w''(1) = 0 \right\}$ *and for every* $d \in C^2(\Re_+)$ *with* $d(0) = u_{1,0}(0)$, *there exists a unique pair of mappings* $u_1 \in C^0(\Re_+ \times [0,1]) \cap C^1((0,+\infty) \times [0,1])$, $u_2 \in C^1(\Re_+ \times [0,1])$ *with* $u_1[t] \in C^2([0,1])$ *for* $t > 0$ *satisfying (4), (5), (7).*

We next give conditions for ISS in the sup norm for system (4), (5).

Theorem 2.2 (ISS in the spatial sup norm): *Consider the hyperbolic-parabolic system (4), (5), where* $K, r, \tilde{a} \in \Re$, $\tilde{b} > 0$ *are constants. Suppose that* $|r\tilde{a}| < K + \pi^2$. *Then there exist constants* $M, \gamma, \delta > 0$ *such that for every* $u_{1,0} \in C^0([0,1])$ *with* $u_{1,0}(1) = 0$, $u_{2,0} \in C^1([0,1])$ *and for every disturbance input* $d \in C^0(\Re_+)$ *with* $d(0) = u_{1,0}(0)$, *for which there exists a unique pair of mappings* $u_1 \in C^0(\Re_+ \times [0,1]) \cap C^1((0,+\infty) \times [0,1])$, $u_2 \in C^1(\Re_+ \times [0,1])$ *with* $u_1[t] \in C^2([0,1])$ *for* $t > 0$ *satisfying (4), (5), (7), the following inequality holds for all* $t \geq 0$*:*

$$\|u_1[t]\|_\infty + \|u_2[t]\|_\infty \leq M \exp(-\delta t)\left(\|u_{1,0}\|_\infty + \|u_{2,0}\|_\infty\right) + \gamma \max_{0 \leq s \leq t}\left(|d(s)|\right) \tag{8}$$

Theorem 2.2 is proved by applying the small-gain methodology. The stability condition $|r\tilde{a}| < K + \pi^2$ is sharp: when $r\tilde{a} \geq K + \pi^2$ there exist solutions of (4), (5) which do not tend to zero. An interpretation of (8) can be made by looking at the wave equation (6) with Kelvin-Voigt damping that corresponds to the hyperbolic-parabolic system (4), (5). In this case, condition (8) implies that a possible anti-stiffness does not dominate the strain. More specifically, for the wave equation with Kelvin-Voigt damping and viscous damping we obtain the following corollary.

Corollary 2.3 (ISS in the spatial sup norm): *Consider the wave equation with Kelvin-Voigt and viscous damping*

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}(t,z) - \sigma \frac{\partial^2 u}{\partial z^2}(t,z)\right) = c^2 \frac{\partial^2 u}{\partial z^2}(t,z) - \mu \frac{\partial u}{\partial t}(t,z), \text{ for } (t,z) \in (0,+\infty)\times(0,1) \tag{9}$$

$$u(t,0) - d(t) = u(t,1) = 0, \text{ for } t \geq 0 \tag{10}$$

where $\sigma, c > 0$, $\mu \geq 0$ *are constants. Suppose that*

$$2c^2 < 2\mu\sigma + \sigma^2\pi^2 . \tag{11}$$

Then there exist constants $M, \gamma, \delta > 0$ *such that for every* $u_0 \in C^2([0,1])$, $w_0 \in C^0([0,1])$ *with* $u_0(1) = 0$, $(w_0 - \sigma u_0'') \in C^1([0,1])$ *and for every disturbance input* $d \in C^0(\Re_+)$ *with* $d(0) = u_0(0)$, *for which there exists a unique mapping* $u \in C^0(\Re_+ \times [0,1]) \cap C^1((0,+\infty)\times[0,1])$ *with* $u[t] \in C^2([0,1])$ *for* $t \geq 0$, *satisfying* $\left(\frac{\partial u}{\partial t} - \sigma u'' + \mu u - \frac{c^2}{\sigma} u\right) \in C^1(\Re_+ \times [0,1])$, *(9), (10),* $u[0] = u_0$ *and* $\frac{\partial u}{\partial t}[0] = w_0$, *the following inequality holds for all* $t \geq 0$:

$$\|u[t]\|_\infty + \left\|\frac{\partial u}{\partial t}[t] - \sigma u''[t]\right\|_\infty \leq M \exp(-\delta t)\left(\|u_0\|_\infty + \|w_0 - \sigma u_0''\|_\infty\right) + \gamma \max_{0\leq s\leq t}\left(|d(s)|\right) \tag{12}$$

It is interesting to notice that the coefficient $\gamma > 0$ appearing in the estimate

$$\|u[t]\|_\infty \leq M \exp(-\delta t)\left(\|u_0\|_\infty + \|w_0 - \sigma u_0''\|_\infty\right) + \gamma \max_{0\leq s\leq t}\left(|d(s)|\right)$$

can be interpreted as the magnification factor of a boundary oscillation to the main body of a string, which has the other end pinned down. Due to the fact that $u(t,0) = d(t)$, the coefficient $\gamma > 0$ appearing above is always greater or equal to 1. An estimate of the magnification factor $\gamma > 0$ can be obtained by following the proof of Theorem 2.2. Indeed, it is shown that under assumption (11), for every $\theta \in \left(0, \frac{\pi}{2}\right)$ and for every $\varepsilon > 0$ with $\frac{(1+\varepsilon)^2|\sigma\mu - c^2|}{\sigma\mu - c^2 + \sigma^2(\pi - 2\theta)^2} < 1$, there exist constants $M, \delta > 0$ such that the following estimate holds for all $t \geq 0$:

$$\|u[t]\|_\infty \leq M \exp(-\delta t)\left(\|u_0\|_\infty + \|w_0 - \sigma u_0''\|_\infty\right) + \frac{1+\varepsilon}{\sin(\theta)\left(1 - (1+\varepsilon)\sqrt{P(\theta)}\right)^2} \max_{0\leq s\leq t}\left(|d(s)|\right)$$

where $P(\theta) := \frac{|\sigma\mu - c^2|}{\sigma\mu - c^2 + \sigma^2(\pi - 2\theta)^2}$.

Turning back to the application that motivated the study of system (4), (5) and using the definitions of $\tilde{a}$, $\tilde{b}$, r and K, we conclude that the stability condition $|r\tilde{a}| < K + \pi^2$ holds automatically for the problem of the underground movement of chemicals.

3. Combination Feedback of Boundary and In-Domain Interconnection

Consider the following system of PDEs for $(t,z)\in(0,+\infty)\times(0,1)$

$$\frac{\partial u_1}{\partial t}(t,z)-p\frac{\partial^2 u_1}{\partial z^2}(t,z)-a\,u_1(t,z)-\int_0^1 b(z,s)\,u_2(t,s)ds=\frac{\partial u_2}{\partial t}(t,z)+c\frac{\partial u_2}{\partial z}(t,z)=0 \tag{13}$$

$$u_1(t,0)=\frac{\partial u_1}{\partial z}(t,1)-qu_1(t,1)=u_2(t,0)-ku_1(t,1)=0\,, \tag{14}$$

where $p,c>0$, $q<1$, $a,k\in\Re$ are constants and $b\in C^1\left([0,1]^2\right)$ is a given function. Model (13), (14) is a system of a parabolic PDE which is interconnected with a first-order hyperbolic PDE, by means of two different terms: the in-domain, non-local term that appears in the parabolic PDE and the boundary non-local trace term $ku_1(t,1)$ that appears in the boundary condition (14). Model (13), (14) is also closely related to a parabolic PDE with delays. Indeed, solving the hyperbolic PDE in (13), we obtain the following parabolic PDE with distributed delays:

$$\frac{\partial u_1}{\partial t}(t,z)=p\frac{\partial^2 u_1}{\partial z^2}(t,z)+a\,u_1(t,z)+ck\int_{t-c^{-1}}^{t} b(z,c(t-s))\,u_1(s,1)ds \tag{15}$$

In what follows, we denote by $b_1\in(-\pi/2,\pi/2)$ the unique solution on $(-\pi/2,\pi/2)$ of the equation $\left(\frac{\pi}{2}-b_1\right)\cot\left(\frac{\pi}{2}-b_1\right)=q$. We next provide existence/uniqueness results for the initial-boundary value (13), (14) with (7), where $u_{1,0},u_{2,0}$ are functions on $[0,1]$.

Theorem 3.1: *Define* $D:=\left\{u\in H^2(0,1):u(0)=u'(1)-qu(1)=0\right\}$. *Consider the initial-boundary value problem (13), (14), (7), where* $p,c>0$, $q<1$, $a,k\in\Re$ *are constants and* $b\in C^1\left([0,1]^2\right)$ *is a given function and suppose that* $p(\pi-2b_1)^2>4a$. *Then for every* $u_{1,0}\in D$, $u_{2,0}\in C^1([0,1])$ *with* $u_{2,0}(0)=ku_{1,0}(1)$, $\theta_{1,0}\in D$ $u'_{2,0}(0)=-c^{-1}k\theta_{1,0}(1)$, *where* $\theta_{1,0}(z)=pu''_{1,0}(z)+a\,u_{1,0}(z)+\int_0^1 b(z,s)\,u_{2,0}(s)ds$ *for* $z\in[0,1]$, *there exists a unique pair of mappings* $u_1\in C^0(\Re_+\times[0,1])\cap C^1((0,+\infty)\times[0,1])$, $u_2\in C^1(\Re_+\times[0,1])$ *with* $u_1[t]\in C^2([0,1])$ *for* $t>0$ *satisfying (13), (14), (7).*

We next provide sufficient conditions for exponential stability in the sup norm for system (13), (14), which are independent of $c>0$. We first need the following auxiliary lemma.

Lemma 3.2: *Suppose that* $p(\pi-2b_1)^2>4a$. *Then there exist constants* $\theta\in(0,\pi)$, $\omega\in[0,\pi-\theta)$ *with* $\omega\cot(\omega+\theta)>q$ *and* $p\omega^2>a$.

Theorem 3.3 (Exponential Stability in the sup norm independent of $c>0$): *Consider the hyperbolic-parabolic system (13), (14), where* $p,c>0$, $q<1$, $a,k\in\Re$ *are constants and* $b\in C^1\left([0,1]^2\right)$ *is a given function and suppose that* $p(\pi-2b_1)^2>4a$. *Moreover, suppose that*

$$|k|\max_{0\le z\le 1}\left(\frac{\sin(\omega+\theta)}{\sin(\omega z+\theta)}\int_0^1|b(z,s)|ds\right)<p\,\omega^2-a \tag{16}$$

for certain constants $\theta\in(0,\pi)$, $\omega\in[0,\pi-\theta)$ *with* $p\omega^2>a$ *and* $\omega\cot(\omega+\theta)>q$, *whose existence is established by Lemma 3.2. Then there exist constants* $M,\delta>0$ *such that for every* $u_{1,0},u_{2,0}\in C^1([0,1])$ *with* $u_{1,0}(1)=u'_{1,0}(1)-qu_{1,0}(1)=u_{2,0}(0)-ku_{1,0}(1)=0$, *for which there exists a unique pair of mappings*

$u_1 \in C^0(\Re_+ \times [0,1]) \cap C^1((0,+\infty)\times[0,1])$, $u_2 \in C^1(\Re_+ \times [0,1])$ with $u_1[t] \in C^2([0,1])$ for $t>0$ satisfying (13), (14), (7), the following inequality holds for all $t \ge 0$:

$$\|u_1[t]\|_\infty + \|u_2[t]\|_\infty \le M \exp(-\delta t)\left(\|u_{1,0}\|_\infty + \|u_{2,0}\|_\infty\right) \tag{17}$$

Inequality (16) imposes a bound on the product of the static gains of the interconnecting, nonlocal terms that may lead the solution far from equilibrium. In the context of the related delay parabolic PDE (15), condition (16) is a delay-independent stability condition. Another use of the stability condition (16) can be shown by studying a crucial question that often arises in engineering practice: is it safe to ignore diffusion? The following example shows how the control engineer can exploit the stability condition (16) in order to answer the question regarding robustness to diffusion.

Example (Is It Safe to Ignore Diffusion?): Applying backstepping to a pair of hyperbolic PDEs (see [1,4]), we obtain the following hyperbolic PDE-PDE loop

$$\frac{\partial w_1}{\partial t}(t,z) + v\frac{\partial w_1}{\partial z}(t,z) = \frac{\partial w_2}{\partial t}(t,z) + c\frac{\partial w_2}{\partial z}(t,z) = 0, \text{ for } (t,z) \in (0,+\infty)\times(0,1) \tag{18}$$

$$w_1(t,0) = w_2(t,0) - k w_1(t,1) = 0, \text{ for } t \ge 0 \tag{19}$$

as the closed-loop system in the transformed variables, where $v,c>0$, $k \in \Re$ are constants. The equilibrium point $0 \in \left(C^0([0,1])\right)^2$ is finite-time stable. However, when diffusion phenomena are present in one of the PDEs, then the actual closed-loop system is described by the equations

$$\frac{\partial w_1}{\partial t}(t,z) - p\frac{\partial^2 w_1}{\partial z^2}(t,z) + v\frac{\partial w_1}{\partial z}(t,z) - p\int_0^1 l(z,s)w_2(t,s)ds = \frac{\partial w_2}{\partial t}(t,z) + c\frac{\partial w_2}{\partial z}(t,z) = 0 \tag{20}$$

$$w_1(t,0) = \frac{\partial w_1}{\partial z}(t,1) = w_2(t,0) - k\, w_1(t,1) = 0, \text{ for } t \ge 0 \tag{21}$$

where $p>0$ is a constant and $l \in C^1\left([0,1]^2\right)$ is a given function. Does exponential stability for system (18), (19) guarantee exponential stability for system (20), (21) when $p>0$ is sufficiently small? The answer is "yes". To prove this, we perform the transformation for $t \ge 0$, $z \in [0,1]$

$$u_1(t,z) = \exp(-vz/(2p))\, w_1(t,z) \quad , \quad u_2(t,z) = w_2(t,z) \tag{22}$$

which transforms system (20), (21) to (13), (14) with $q := -v/(2p)$, $b(z,s) = p\exp(-vz/(2p))l(z,s)$ for $z,s \in [0,1]$ and $a := -v^2/(4p)$. Applying Theorem 3.3 with $\theta = \pi/2$, $\omega = 0$, we conclude that $0 \in \left(C^0([0,1])\right)^2$ is exponentially stable in the sup norm for system (20), (21), provided that $2p\sqrt{|k| \max_{0 \le z \le 1}\left(\int_0^1 |l(z,s)|ds\right)} < v$.

Consequently, there exists $P \in (0,+\infty]$ such that $0 \in \left(C^0([0,1])\right)^2$ is exponentially stable in the sup norm for system (20), (21), provided that $p \in (0,P)$. $\triangleleft$

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